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Noncommutative instantons in higher dimensions, vortices and topological K-cycles

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Abstract: We construct explicit BPS and non-BPS solutions of the $\mathrm{U}(2k)$ Yang-Mills equations on the noncommutative space $\mathbb{R}^{2n}_\theta \times S^2$ with finite energy and topological charge. By twisting with a Dirac multi-monopole bundle over S^2, we reduce the Donaldson-Uhlenbeck-Yau equations on $\mathbb{R}^{2n}_\theta \times S^2$ to vortex-type equations for a pair of $\mathrm{U}(k)$ gauge fields and a bi-fundamental scalar field on $\mathbb{R}^{2n}_\theta$. In the $\mathrm{SO}(3)$-invariant case the vortices on $\mathbb{R}^{2n}_\theta$ determine multi-instantons on $\mathbb{R}^{2n}_\theta \times S^2$. We show that these solutions give natural physical realizations of Bott periodicity and vector bundle modification in topological K-homology, and can be interpreted as a blowing-up of D0-branes on $\mathbb{R}^{2n}_\theta$ into spherical D2-branes on $\mathbb{R}^{2n}_\theta \times S^2$. In the generic case with broken rotational symmetry, we argue that the D0-brane charges on $\mathbb{R}^{2n}_\theta \times S^2$ provide a physical interpretation of the Adams operations in K-theory.

Keywords: Solitons Monopoles and Instantons, D-branes, Non-Commutative Geometry, Integrable Field Theories.

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Contents

1. Introduction

It is a major enterprise of mathematical physics to try to extend the success story of gauge field theory by adding supersymmetry or extra dimensions, by embedding it into string theory, by deforming it noncommutatively, or by combinations of these ideas. Specifically, Grönewold-Moyal type noncommutative deformations have been the subject of intense research in recent years [1, 2]. Besides settling the issue of perturbative quantization, it is important to map out the classical configuration space of noncommutative gauge theories and to characterize its role in string theory. All celebrated BPS configurations, such as instantons [3], monopoles [4] and vortices [5, 6], have been generalized to the noncommutative case, originally in [7], [8] and [9, 10],[1] respectively (see [13] for a recent review and references). The description of D-branes as solitons in open string field theory simplifies dramatically in the context of a noncommutative tachyon field theory, with the D-branes appear-

[1]For related works on flux tube solutions see [11, 12].

ing as noncommutative solitons [14]. The relation between D-branes and noncommutative tachyons makes manifest [15, 16] the relationship between D-branes and K-theory [17, 18].

In the superstring theory context, one encounters gauge theories in spacetime dimensionalities up to ten. Already 20 years ago, BPS-type equations in more than four dimensions were proposed [19, 20] and their solutions investigated e.g. in [20, 21]. More recently, noncommutative instantons in higher dimensions and their brane interpretations have been considered in [22]–[25]. For nonabelian gauge theory on a Kähler manifold the most natural BPS condition lies in the Donaldson-Uhlenbeck-Yau equations [26, 27], which generalize the four-dimensional self-duality equations.

In this paper we investigate the Donaldson-Uhlenbeck-Yau equations on the noncommutative spaces $\mathbb{R}^{2n}_\theta \times S^2$ for the gauge group $\mathrm{U}(2k)$. By employing a reduction via a Dirac monopole bundle over the S^2 we obtain generalized coupled vortex equations on $\mathbb{R}^{2n}_\theta$ for a pair of $\mathrm{U}(k)$ gauge fields and a complex matrix-valued scalar field. Invoking partial isometries and the ABS construction, BPS and also non-BPS solutions are found, which are labelled by three integers and carry a number of moduli. We calculate the topological charge and energy of these "multi-instanton" solutions and mention some extremal cases. We then address the problem of assigning K-theory classes to the explicit noncommutative instanton solutions that are found. We will find that in the simplest instance of a one-monopole configuration on the S^2 our solutions provide physical realizations of the vector bundle modification relation in topological K-homology, and thereby yield explicit K-cycle representatives. For multi-monopole configurations on the S^2, we argue that the solutions instead realize symmetry operations in K-theory. Using these correspondences we argue that in the former case our instanton solutions describe D2-branes on $\mathbb{R}^{2n}_\theta \times S^2$ which are equivalent to D0-branes on $\mathbb{R}^{2n}_\theta$ described by vortices. In the latter case, the brane interpretations are less transparent, and we argue that the vortex solutions represent D0-branes on $\mathbb{R}^{2n}_\theta \times S^2$ which contain residual moduli from their locations inside the S^2.

The organisation of this paper is as follows. In section 2 we recall various aspects of the noncommutative space $\mathbb{R}^{2n}_\theta \times S^2$, and in section 3 we write down the Donaldson-Uhlenbeck-Yau equations on it. In section 4 we describe our particular ansatz that we use to solve these equations, and show in section 5 how together they reduce to vortex equations on $\mathbb{R}^{2n}_\theta$. Sections 6 and 7 then deal with explicit BPS and non-BPS solutions, respectively, to the Yang-Mills equations on $\mathbb{R}^{2n}_\theta \times S^2$. In section 8 the topological charge of these configurations is computed, and the Yang-Mills action is evaluated on them in section 9. In section 10 solutions in the zero monopole and zero tachyon sectors are explicitly constructed. In section 11 we show that our solutions naturally define K-cycles, and this correspondence is exploited in section 12 to assign D0-brane charges to them. Finally, in section 13 we summarize our findings and make some further remarks concerning the brane interpretations of the multi-instanton solutions on $\mathbb{R}^{2n}_\theta \times S^2$.

2. The noncommutative space $\mathbb{R}^{2n}_\theta \times S^2$

Geometry of $\mathbb{R}^{2n} \times S^2$. In order to set the stage for the Donaldson-Uhlenbeck-Yau equations and their instanton solutions, we begin with the (commutative) manifold $\mathbb{R}^{2n} \times S^2$

carrying the riemannian metric

$$\mathrm{d}s^2 = \delta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + R^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right) = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \tag{2.1}$$

where $\mu,\nu = 1,\ldots,2n$ but $i,j = 1,\ldots,2n{+}2$, and $x = (x^\mu)$ are coordinates on $\mathbb{R}^{2n}$ while $x^{2n+1}{=}\vartheta$ and $x^{2n+2}{=}\varphi$ parametrize the standard two-sphere S^2 of constant radius R, i.e. $0 \le \varphi \le 2\pi$ and $0 \le \vartheta \le \pi$. We use the Einstein summation convention for repeated indices. The volume two-form on S^2 reads

$$\sqrt{\det(g_{ij})}\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi =: \omega_{\vartheta\varphi}\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi = \omega \qquad \Longrightarrow \qquad \omega_{\vartheta\varphi} = -\omega_{\varphi\vartheta} = R^2\sin\vartheta\,. \tag{2.2}$$

The manifold $\mathbb{R}^{2n}\times S^2$ is Kähler, with local complex coordinates $z^1,\ldots,z^n,y$ where

$$z^a = x^{2a-1} - \mathrm{i}\,x^{2a} \qquad \text{and} \qquad \bar z^{\bar a} = x^{2a-1} + \mathrm{i}\,x^{2a} \qquad \text{with } a = 1,\ldots,n \tag{2.3}$$

and

$$y = \frac{R\,\sin\vartheta}{(1+\cos\vartheta)}\,\exp\left(-\mathrm{i}\varphi\right)\,, \qquad \bar y = \frac{R\,\sin\vartheta}{(1+\cos\vartheta)}\,\exp\left(\mathrm{i}\varphi\right)\,, \tag{2.4}$$

so that $1{+}\cos\vartheta = \frac{2R^2}{R^2+y\bar y}$. In these coordinates, the metric takes the form

$$\mathrm{d}s^2 = \delta_{a\bar b}\,\mathrm{d}z^a\,\mathrm{d}\bar z^{\bar b} + \frac{4\,R^4}{(R^2+y\bar y)^2}\,\mathrm{d}y\,\mathrm{d}\bar y \tag{2.5}$$

with $\delta_{a\bar a}{=}\delta^{a\bar a}{=}1$ (and all other entries vanishing), and the Kähler two-form reads

$$\Omega = -\frac{\mathrm{i}}{2}\left\{\delta_{a\bar b}\,\mathrm{d}z^a\wedge\mathrm{d}\bar z^{\bar b} + \frac{4\,R^4}{(R^2+y\bar y)^2}\,\mathrm{d}y\wedge\mathrm{d}\bar y\right\} = -\frac{\mathrm{i}}{2}\,\delta_{a\bar b}\,\mathrm{d}z^a\wedge\mathrm{d}\bar z^{\bar b} + \omega_{\vartheta\varphi}\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi\,. \tag{2.6}$$

For later use, we also note here the derivatives

$$\partial_{z^a} = \frac{1}{2}\left(\partial_{2a-1} + \mathrm{i}\partial_{2a}\right) \qquad \text{and} \qquad \partial_{\bar z^{\bar a}} = \frac{1}{2}\left(\partial_{2a-1} - \mathrm{i}\partial_{2a}\right)\,, \tag{2.7}$$

where $\partial_\mu := \partial/\partial x^\mu$ for $\mu{=}1,\ldots,2n$.

Noncommutative deformation. Let us now pass to a noncommutative deformation of the flat part of the manifold under consideration, i.e. $\mathbb{R}^{2n}\times S^2 \to \mathbb{R}^{2n}_\theta\times S^2$. Note that the S^2 factor remains commutative in this paper. As is well known, classical field theory on the noncommutative space $\mathbb{R}^{2n}_\theta$ may be realized in either a star-product formulation or in an operator formalism. While the first approach alters the product of functions on $\mathbb{R}^{2n}$, the second one turns these functions f into linear operators $\hat f$ acting on the n-harmonic oscillator Fock space $\mathcal{H}$. The noncommutative space $\mathbb{R}^{2n}_\theta$ may then be defined by declaring its coordinate functions $\hat x^1,\ldots,\hat x^{2n}$ to obey the Heisenberg algebra relations

$$[\hat x^\mu\,,\,\hat x^\nu] = \mathrm{i}\,\theta^{\mu\nu} \tag{2.8}$$

with a constant antisymmetric tensor $\theta^{\mu\nu}$. The coordinates can be chosen in such a way that the matrix $\theta = (\theta^{\mu\nu})$ is block-diagonal with non-vanishing components

$$\theta^{2a-1\,2a} = -\theta^{2a\,2a-1} =: \theta^a\,. \tag{2.9}$$

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We will assume that all $\theta^a \geq 0$, as the general case does not hide additional complications. For the noncommutative version of the complex coordinates (2.3) we have

$$\left[\hat{z}^a\,,\,\hat{\bar{z}}^{\bar{b}}\right] = -2\,\delta^{a\bar{b}}\,\theta^a =: \theta^{a\bar{b}} = -\theta^{\bar{b}a} \leq 0\,, \qquad \text{and all other commutators vanish}. \tag{2.10}$$

The Fock space $\mathcal{H}$ is spanned by the basis states

$$|k_1,\ldots,k_n\rangle = \prod_{a=1}^{n} (2\theta^a\,k_a!)^{-1/2}\,(\hat{z}^a)^{k_a}|0,\ldots,0\rangle \qquad \text{for } k_a = 0,1,2,\ldots\,, \tag{2.11}$$

which are connected by the action of creation and annihilation operators subject to the commutation relations

$$\left[\frac{\hat{\bar{z}}^{\bar{b}}}{\sqrt{2\theta^b}}\,,\,\frac{\hat{z}^a}{\sqrt{2\theta^a}}\right] = \delta^{a\bar{b}}\,. \tag{2.12}$$

We recall that, in the Weyl operator realization $f\mapsto\hat{f}$, derivatives of a function f on $\mathbb{R}^{2n}$ get mapped according to

$$\widehat{\partial_{z^a} f} = \theta_{a\bar{b}}\left[\hat{\bar{z}}^{\bar{b}}\,,\,\hat{f}\,\right] =: \partial_{\hat{z}^a}\hat{f} \qquad \text{and} \qquad \widehat{\partial_{\bar{z}^{\bar{a}}} f} = \theta_{\bar{a}b}\left[\hat{z}^b\,,\,\hat{f}\,\right] =: \partial_{\hat{\bar{z}}^{\bar{a}}}\hat{f}\,, \tag{2.13}$$

where $\theta_{a\bar{b}}$ is defined via $\theta_{b\bar{c}}\,\theta^{\bar{c}a} = \delta^a_b$ so that $\theta_{a\bar{b}} = -\theta_{\bar{b}a} = \frac{\delta_{a\bar{b}}}{2\theta^a}$. Finally, there is the relation

$$\int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}x\;f(x) = \left(\prod_{a=1}^{n} 2\pi\theta^a\right)\mathrm{Tr}_{\mathcal{H}}\;\hat{f}\,. \tag{2.14}$$

Taking the product of $\mathbb{R}^{2n}_\theta$ with the commutative sphere S^2 means extending the noncommutativity matrix θ by vanishing entries in the two new directions. A more detailed description of noncommutative field theories can be found in [2].

3. The Donaldson-Uhlenbeck-Yau equations

The generalization of the four-dimensional self-duality equations to higher dimensions is not unique. A particularly natural extension is given by the Donaldson-Uhlenbeck-Yau (DUY) equations [26, 27] which can be formulated on any Kähler manifold. Their importance derives from the BPS property, i.e. they yield stable solutions of the Yang-Mills equations. We shall present the DUY equations first in generality, then on $\mathbb{R}^{2n}\times S^2$, and finally on $\mathbb{R}^{2n}_\theta\times S^2$.

Let M_{2q} be a complex $q{=}n{+}1$ dimensional Kähler manifold with some local real coordinates (x^i) and a tangent space basis $\partial_i := \partial/\partial x^i$ for $i,j = 1,\ldots,2q$, so that the metric and Kähler two-form read $\mathrm{d}s^2 = g_{ij}\;\mathrm{d}x^i\;\mathrm{d}x^j$ and $\Omega = \Omega_{ij}\;\mathrm{d}x^i\wedge\mathrm{d}x^j$, respectively. Consider a rank $2k$ complex vector bundle over M_{2q} with a chosen gauge potential $\mathcal{A} = \mathcal{A}_i\;\mathrm{d}x^i$. The curvature two-form $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$ has components $\mathcal{F}_{ij} = \partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i + [\mathcal{A}_i,\mathcal{A}_j]$ and the Kähler decomposition $\mathcal{F} = \mathcal{F}^{2,0} + \mathcal{F}^{1,1} + \mathcal{F}^{0,2}$. Both $\mathcal{A}_i$ and $\mathcal{F}_{ij}$ take values in the Lie algebra $u(2k)$. The DUY equations [26, 27] on M_{2q} are

$$*\Omega\wedge\mathcal{F} = 0 \qquad \text{and} \qquad \mathcal{F}^{0,2} = 0 = \mathcal{F}^{2,0}\,, \tag{3.1}$$

where $*$ is the Hodge duality operator. In our local coordinates (x^i) we have $q!\,(*\Omega \wedge \mathcal{F}) = (\Omega, \mathcal{F})\,\Omega^q = \Omega^{ij}\mathcal{F}_{ij}\,\Omega^q$ where Ω^{ij} are defined via $\Omega^{ij}\,\Omega_{jl} = \delta^i_l$. For q=2 the DUY equations (3.1) coincide with the anti-self-dual Yang-Mills (ASDYM) equations

$$*\mathcal{F} = -\mathcal{F} \tag{3.2}$$

introduced in [3].

Specializing now M_{2q} to be $\mathbb{R}^{2n} \times S^2$, the DUY equations (3.1) in the complex local coordinates (z^a, y) take the form[2]

$$\delta^{a\bar{b}}\,\mathcal{F}_{z^a\bar{z}^{\bar{b}}} + \frac{(R^2+y\bar{y})^2}{4\,R^4}\,\mathcal{F}_{y\bar{y}} = 0\,, \tag{3.3}$$

$$\mathcal{F}_{\bar{z}^{\bar{a}}\bar{z}^{\bar{b}}} = 0 = \mathcal{F}_{z^a z^b}\,, \tag{3.4}$$

$$\mathcal{F}_{\bar{z}^{\bar{a}}\bar{y}} = 0 = \mathcal{F}_{z^a y}\,, \tag{3.5}$$

where $a, b = 1, \ldots, n$. Using the formulae (2.4), we obtain

$$\begin{aligned} \mathcal{F}_{\bar{z}^{\bar{a}}\bar{y}} &= \mathcal{F}_{\bar{z}^{\bar{a}}\vartheta}\,\frac{\partial\vartheta}{\partial\bar{y}} + \mathcal{F}_{\bar{z}^{\bar{a}}\varphi}\,\frac{\partial\varphi}{\partial\bar{y}} = \frac{1}{\bar{y}}\,\left(\sin\vartheta\,\mathcal{F}_{\bar{z}^{\bar{a}}\vartheta} - \mathrm{i}\,\mathcal{F}_{\bar{z}^{\bar{a}}\varphi}\right) = -(\mathcal{F}_{z^a y})^\dagger\,, \\ \mathcal{F}_{y\bar{y}} &= \mathcal{F}_{\vartheta\varphi}\left|\frac{\partial(\vartheta,\varphi)}{\partial(y,\bar{y})}\right| = \frac{1}{2\mathrm{i}}\,\frac{\sin\vartheta}{y\bar{y}}\,\mathcal{F}_{\vartheta\varphi} = \frac{1}{2\mathrm{i}}\,\frac{(1+\cos\vartheta)^2}{R^2\sin\vartheta}\,\mathcal{F}_{\vartheta\varphi}\,, \end{aligned} \tag{3.6}$$

and one can write the DUY equations on $\mathbb{R}^{2n} \times S^2$ in the alternative form

$$2\mathrm{i}\,\delta^{a\bar{b}}\,\mathcal{F}_{z^a\bar{z}^{\bar{b}}} + \frac{1}{R^2\sin\vartheta}\,\mathcal{F}_{\vartheta\varphi} = 0\,, \qquad \mathcal{F}_{\bar{z}^{\bar{a}}\bar{z}^{\bar{b}}} = 0\,, \qquad \sin\vartheta\,\mathcal{F}_{\bar{z}^{\bar{a}}\vartheta} - \mathrm{i}\,\mathcal{F}_{\bar{z}^{\bar{a}}\varphi} = 0\,, \tag{3.7}$$

along with their hermitian conjugates. It is easy to show that any solution of these $n(n+1)+1$ equations also satisfies the full Yang-Mills equations.

The transition to the noncommutative DUY equations is trivially achieved by going over to operator-valued objects everywhere. In particular, the field strength components in (3.3)–(3.7) then read $\hat{\mathcal{F}}_{ij} = \partial_{\hat{x}^i}\hat{\mathcal{A}}_j - \partial_{\hat{x}^j}\hat{\mathcal{A}}_i + [\hat{\mathcal{A}}_i, \hat{\mathcal{A}}_j]$, where $\hat{\mathcal{A}}_i$ are simultaneously $u(2k)$ and operator valued. To avoid a cluttered notation, we drop the hats from now on.

4. Ansatz for the gauge potential

Rather than attempting to solve the DUY equations (3.7) in full generality, we shall prescribe a specific S^2 dependence for the gauge potential $\mathcal{A}$, generalizing an ansatz due to Taubes [6]. His ansatz[3] was introduced for SU(2) gauge fields on $\mathbb{R}^2 \times S^2$ and reduced the ASDYM equations (3.2) to the vortex equations on $\mathbb{R}^2$. Our ansatz will eliminate the spherical coordinates and dimensionally reduce the U($2k$) DUY equations on $\mathbb{R}^{2n}_\theta \times S^2$ to generalized coupled vortex equations for a pair of U(k) gauge fields and a GL($k, \mathbb{C}$)-valued scalar field living on $\mathbb{R}^{2n}_\theta$, to be described in the next section.

[2]Note that these equations are not integrable even for $n = 1$. Therefore, neither the dressing nor splitting approaches developed in [28] for equations on noncommutative spaces can be applied here. The modified ADHM construction [7] also does not work in this case.

[3]Similarly, Witten's ansatz [29] for gauge fields on $\mathbb{R}^4$ reduces (3.2) to the vortex equations on the hyperbolic space H^2 (cf. [30] for noncommutative $\mathbb{R}^4$).

Choice of ansatz. Our ansatz is motivated by imposing invariance under the SO(3) isometry group of the two-sphere. Naively, this would seem to lead to the trivial reduction

$$\mathcal{A}(x,\vartheta,\varphi) = \mathcal{A}_\mu(x)\ \mathrm{d}x^\mu\,. \tag{4.1}$$

However, it is natural to allow for gauge transformations to accompany the SO(3) action [31], and so some "twisting" can occur in the reduction. It is known that the angular dependence in this case is determined via an embedding of the Dirac monopole (or anti-monopole) U(1) bundle $\mathcal{L}_{\mathrm{D}} \to S^2$ into a trivial SU(2) bundle over S^2 [6]. In the SU(2) Wu-Yang description [32], this monopole is encoded in the matrix

$$Q(\vartheta,\varphi) := \mathrm{i}\vec{\sigma}\cdot\vec{n}(\vartheta,\varphi) \qquad \text{with } \vec{n}^2 = 1\,, \tag{4.2}$$

where $\vec{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ denotes the Pauli spin matrices and $\vec{n}(\vartheta,\varphi)$ parametrizes the unit two-sphere in three-dimensional space. In this paper we will consider a more general ansatz by admitting Dirac *multi*-monopoles. More concretely, we take m Dirac monopoles sitting on top of each other ($m \in \mathbb{Z}$) as our angular configuration.[4] Note that this is no longer SO(3) invariant for $|m| > 1$. Nevertheless, this generalization is easily performed by choosing [33, 34]

$$Q = Q(\vartheta, m\varphi) = \mathrm{i}\,(\sin\vartheta\cos(m\varphi)\,\sigma_1 \ + \ \sin\vartheta\sin(m\varphi)\ \sigma_2 \ + \ \cos\vartheta\ \sigma_3) = -Q^\dagger\,, \quad m\in\mathbb{Z}\,. \tag{4.3}$$

The $su(2)$ valued m-monopole connection and curvature read [33, 34]

$$\mathcal{A}_{su(2)} = -\frac{1}{2}Q\ \mathrm{d}Q \qquad \text{and} \qquad \mathcal{F}_{su(2)} = -\frac{1}{4}\ \mathrm{d}Q\wedge\mathrm{d}Q\,. \tag{4.4}$$

The simplest way to embed the $su(2)$ matrix Q into $u(2k)$ consists of the reduction $u(2k) \to u(k)\otimes\mathrm{i}Q$. This yields

$$\mathcal{A}(x,\vartheta,\varphi) = \frac{1}{2}\,A_\mu(x)\ \mathrm{d}x^\mu\otimes\mathrm{i}Q + \frac{1}{2}\,A_Q(x)\otimes\mathrm{i}Q\ \mathrm{d}Q\,, \tag{4.5}$$

where A_μ and A_Q take values in $u(k)$. Since noncommutative gauge transformations need the full $u(k)\oplus u(k)$ for closure we should add to this ansatz pieces from $u(k)\otimes\mathbf{1}_2$. Altogether, it implies that our ansatz for the $u(2k)$-valued gauge potential $\mathcal{A}$ becomes

$$\mathcal{A}(x,\vartheta,\varphi) = \frac{1}{2}\left\{A\,\mathrm{i}Q + B\,\mathbf{1}_2 + (\phi_1 - 1)\,Q\ \mathrm{d}Q + \phi_2\ \mathrm{d}Q\right\}, \tag{4.6}$$

where the one-forms $A = A_\mu(x)\ \mathrm{d}x^\mu$ and $B = B_\mu(x)\ \mathrm{d}x^\mu$ take values in $u(k)$ (i.e. they are anti-hermitian), while the scalars $\phi_1 = \phi_1(x)$ and $\phi_2 = \phi_2(x)$ are $k\times k$ hermitian matrices. Note that the fields (A, B, ϕ_1, ϕ_2) do not depend on ϑ or φ but act as operators on the Fock space $\mathcal{H}$ in the noncommutative gauge theory.

[4] We do not turn on the moduli associated with their separation or relative isospin orientation.

Monopole projectors and gauge potential. The $su(2)$ matrix Q satisfies the identities

$$Q^2 = -\mathbf{1}_2 \qquad \Longrightarrow \qquad (Q\,\mathrm{d}Q)^\dagger = \mathrm{d}Q\;Q = -Q\;\mathrm{d}Q\,, \tag{4.7}$$

$$Q\,\frac{\partial Q}{\partial\varphi} = m\,\sin\vartheta\,\frac{\partial Q}{\partial\vartheta} \qquad \Longrightarrow \qquad \frac{\partial Q}{\partial\varphi} = -m\sin\vartheta\;Q\,\frac{\partial Q}{\partial\vartheta}\,, \tag{4.8}$$

$$\mathrm{d}Q\wedge\mathrm{d}Q = \Big(\frac{\partial Q}{\partial\vartheta}\,\frac{\partial Q}{\partial\varphi} - \frac{\partial Q}{\partial\varphi}\,\frac{\partial Q}{\partial\vartheta}\Big)\;\mathrm{d}\vartheta\wedge\mathrm{d}\varphi = -2m\,Q\,\sin\vartheta\;\mathrm{d}\vartheta\wedge\mathrm{d}\varphi\,. \tag{4.9}$$

It is convenient to introduce the hermitian projectors

$$P_+ = \frac{1}{2}\,(\mathbf{1}_2 + \mathrm{i}Q) \qquad \text{and} \qquad P_- = \frac{1}{2}\,(\mathbf{1}_2 - \mathrm{i}Q) \tag{4.10}$$

on $\mathbb{C}^2$ which define the degree $|m|$ monopole and anti-monopole line bundles $(\mathcal{L}_{\mathrm{D}})^m \to S^2$ (with $(\mathcal{L}_{\mathrm{D}})^{-|m|} := (\overline{\mathcal{L}_{\mathrm{D}}})^{|m|}$). They satisfy

$$P_\pm^2 = P_\pm\,, \qquad P_+ + P_- = \mathbf{1}_2\,, \qquad P_+P_- = 0 \qquad \mathrm{tr}_{2\times2}\;P_\pm = 1\,, \tag{4.11}$$

$$\begin{aligned}\mathrm{d}P_\pm = {}& \Big(\big[(1{-}m)\,P_+ + (1{+}m)\,P_-\big]\mathrm{e}^{\mathrm{i}\varphi}\mathrm{d}y + \big[(1{+}m)\,P_+ + (1{-}m)\,P_-\big]\mathrm{e}^{-\mathrm{i}\varphi}\mathrm{d}\bar{y}\Big)\times \\ & \times\frac{1{+}\cos\vartheta}{R}\,\frac{\partial P_\pm}{\partial\vartheta}\,.\end{aligned} \tag{4.12}$$

The corresponding field combinations are

$$A^\pm := \frac{1}{2}\,(B\pm A) \qquad \Longleftrightarrow \qquad A = A^+ - A^-\,, \quad B = A^+ + A^- \tag{4.13}$$

and

$$\phi := \phi_1 + \mathrm{i}\,\phi_2 \qquad \Longleftrightarrow \qquad \phi_1 = \frac{1}{2}\,(\phi+\phi^\dagger)\,, \quad \phi_2 = \frac{\mathrm{i}}{2}\,(\phi^\dagger - \phi)\,. \tag{4.14}$$

In terms of these degrees of freedom, the ansatz (4.6) becomes

$$\mathcal{A} = A^+\,P_+ + A^-\,P_- + (1-\phi)\,P_+\mathrm{d}P_+ + (1-\phi^\dagger)\,P_-\mathrm{d}P_-\,. \tag{4.15}$$

With $A_a^\pm := A_{z^a}^\pm$ and $A_{\bar a}^\pm := A_{\bar z^{\bar a}}^\pm$, this implies

$$\mathcal{A}_a := \mathcal{A}_{z^a} = A_a^+\,P_+ + A_a^-\,P_- = -(\mathcal{A}_{\bar a})^\dagger\,, \tag{4.16}$$

$$\mathcal{A}_y = (1-\phi)\,P_+\,\frac{\partial P_+}{\partial y} + (1-\phi^\dagger)\,P_-\,\frac{\partial P_-}{\partial y} = -(\mathcal{A}_{\bar y})^\dagger\,. \tag{4.17}$$

Field strength tensor. The calculation of the curvature

$$\begin{aligned}\mathcal{F} &= \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = \frac{1}{2}\,\mathcal{F}_{ij}\;\mathrm{d}x^i\wedge\mathrm{d}x^j \\ &= \frac{1}{2}\,\mathcal{F}_{\mu\nu}\;\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu + \mathcal{F}_{\mu\vartheta}\;\mathrm{d}x^\mu\wedge\mathrm{d}\vartheta + \mathcal{F}_{\mu\varphi}\;\mathrm{d}x^\mu\wedge\mathrm{d}\varphi + \mathcal{F}_{\vartheta\varphi}\;\mathrm{d}\vartheta\wedge\mathrm{d}\varphi\end{aligned} \tag{4.18}$$

for $\mathcal{A}$ of the form (4.6) yields

$$\begin{aligned}4\,\mathcal{F} = {}& \big(2\;\mathrm{d}A + A\wedge B + B\wedge A\big)\,\mathrm{i}Q \;+ \big(2\;\mathrm{d}B + A\wedge A + B\wedge B\big)\,\mathbf{1}_2 + \\ & + \big([\phi_1,\phi_2]\,Q + (\phi_1^2+\phi_2^2-1)\,\mathbf{1}_2\big)\mathrm{d}Q\wedge\mathrm{d}Q +\end{aligned} \tag{4.19}$$

$$+\Big(\big(2\,\mathrm{d}\phi_1+\mathrm{i}A\,\phi_2+\mathrm{i}\phi_2\,A+[B,\phi_1]\big)Q+\big(2\,\mathrm{d}\phi_2-\mathrm{i}A\,\phi_1-\mathrm{i}\phi_1\,A+[B,\phi_2]\big)\mathbf{1}_2\Big)\wedge\mathrm{d}Q\,.$$

Rewriting $\mathcal{F}$ in terms of $A^{\pm}$ and $\phi,\phi^{\dagger}$ as in (4.15), we obtain

$$\begin{aligned}\mathcal{F} &= F^+\,P_+ + F^-\,P_- + (1-\phi\,\phi^{\dagger})\,P_+\mathrm{d}P_+\wedge\mathrm{d}P_+ + (1-\phi^{\dagger}\,\phi)\,P_-\mathrm{d}P_-\wedge\mathrm{d}P_- - \\ &\quad -D\phi\wedge P_+\mathrm{d}P_+ - (D\phi)^{\dagger}\wedge P_-\mathrm{d}P_-\,,\end{aligned}\tag{4.20}$$

where $F^{\pm}:=\mathrm{d}A^{\pm}+A^{\pm}\wedge A^{\pm}$ and we have introduced the bi-fundamental covariant derivative

$$D\phi := \mathrm{d}\phi + A^+\,\phi - \phi\,A^-\,.\tag{4.21}$$

With $F^{\pm}_{ab}:=F^{\pm}_{z^az^b}$ and so on, from (4.20) we have

$$\mathcal{F}_{\bar a\bar b} := \mathcal{F}_{\bar z^{\bar a}\bar z^{\bar b}} = F^+_{\bar a\bar b}\,P_+ + F^-_{\bar a\bar b}\,P_- = -(\mathcal{F}_{ab})^{\dagger}\,,\tag{4.22}$$

$$\mathcal{F}_{a\bar b} := \mathcal{F}_{z^a\bar z^{\bar b}} = F^+_{a\bar b}\,P_+ + F^-_{a\bar b}\,P_- = -(\mathcal{F}_{\bar a b})^{\dagger}\,,\tag{4.23}$$

$$\mathcal{F}_{y\bar y} = -\frac{m\,R^2}{(R^2+y\bar y)^2}\Big\{(1-\phi\,\phi^{\dagger})\,P_+ - (1-\phi^{\dagger}\,\phi)\,P_-\Big\} = -(\mathcal{F}_{\bar y y})^{\dagger}\,,\tag{4.24}$$

$$\mathcal{F}_{\bar z^{\bar a}\bar y} = -(\bar D_{\bar a}\phi)\,P_+\,\frac{\partial P_+}{\partial\bar y} - (D_a\phi)^{\dagger}\,P_-\,\frac{\partial P_-}{\partial\bar y} = -(\mathcal{F}_{z^a y})^{\dagger}\,,\tag{4.25}$$

$$\mathcal{F}_{\bar z^{\bar a} y} = -(\bar D_{\bar a}\phi)\,P_+\,\frac{\partial P_+}{\partial y} - (D_a\phi)^{\dagger}\,P_-\,\frac{\partial P_-}{\partial y} = -(\mathcal{F}_{z^a\bar y})^{\dagger}\,.\tag{4.26}$$

Inspecting (4.12) we notice a simplification for $m=1$, namely

$$\frac{\partial P_{\pm}}{\partial\bar y}\propto P_+\,,\qquad \frac{\partial P_{\pm}}{\partial y}\propto P_-\qquad\Longrightarrow\qquad \mathcal{F}_{\bar z^{\bar a}\bar y}\propto(\bar D_{\bar a}\phi)\,P_+\,,\qquad \mathcal{F}_{\bar z^{\bar a}y}\propto(D_a\phi)^{\dagger}\,P_-\,.\tag{4.27}$$

For $m=-1$ one interchanges y and $\bar y$. If $|m|\neq 1$ then both terms are present in (4.25) and (4.26).

5. Generalized coupled vortex equations on $\mathbb{R}^{2n}_{\theta}$

Reduction to $\mathbb{R}^{2n}_{\theta}$. Let us now see what becomes of the DUY equations on $\mathbb{R}^{2n}_{\theta}\times S^2$ for gauge potentials of the form proposed in the previous section. Substituting (4.22)–(4.24) into (3.3) and (3.4), now understood in the noncommutative setting, we obtain

$$\delta^{a\bar b}\,F^+_{a\bar b} = +\frac{m}{4R^2}\,\Big(1-\phi\,\phi^{\dagger}\Big)\qquad\text{and}\qquad F^+_{\bar a\bar b}=0=F^+_{ab}\,,\tag{5.1}$$

$$\delta^{a\bar b}\,F^-_{a\bar b} = -\frac{m}{4R^2}\,\Big(1-\phi^{\dagger}\,\phi\Big)\qquad\text{and}\qquad F^-_{\bar a\bar b}=0=F^-_{ab}\,.\tag{5.2}$$

Setting (4.25) to zero (see (3.5)) leads to the case distinction

$$m=+1\quad\Longrightarrow\quad \bar\partial_{\bar a}\phi + A^+_{\bar a}\,\phi - \phi\,A^-_{\bar a} = 0\tag{5.3}$$

$$m=-1\quad\Longrightarrow\quad \partial_a\phi + A^+_a\,\phi - \phi\,A^-_a = 0\tag{5.4}$$

$$|m|\neq 1\quad\Longrightarrow\quad \bar\partial_{\bar a}\phi + A^+_{\bar a}\,\phi - \phi\,A^-_{\bar a} = 0\qquad\text{and}\qquad \partial_a\phi + A^+_a\,\phi - \phi\,A^-_a = 0\,.\tag{5.5}$$

We shall call (5.1)–(5.5) the generalized coupled vortex equations.

Instead of working with the gauge potentials $A^\pm_\mu$ we shall use the operators $X^\pm_\mu$ defined via

$$X^\pm_a := A^\pm_a + \theta_{a\bar b}\,\bar z^{\bar b} \qquad \text{and} \qquad X^\pm_{\bar a} := A^\pm_{\bar a} + \theta_{\bar a b}\,z^b\,, \tag{5.6}$$

in terms of which the field strength tensor reads

$$F^\pm_{a\bar b} = \left[X^\pm_a\,,\,X^\pm_{\bar b}\right] + \theta_{a\bar b} \qquad \text{and} \qquad F^\pm_{\bar a\bar b} = \left[X^\pm_{\bar a}\,,\,X^\pm_{\bar b}\right]. \tag{5.7}$$

Our generalized coupled vortex equations (5.1)–(5.5) can then be rewritten as

$$\delta^{a\bar b}\left\{\left[X^+_a\,,\,X^+_{\bar b}\right] + \theta_{a\bar b}\right\} = +\frac{m}{4R^2}\,\left(1-\phi\,\phi^\dagger\right) \quad \text{and} \quad \left[X^+_{\bar a}\,,\,X^+_{\bar b}\right] = 0 = \left[X^+_a\,,\,X^+_b\right], \tag{5.8}$$

$$\delta^{a\bar b}\left\{\left[X^-_a\,,\,X^-_{\bar b}\right] + \theta_{a\bar b}\right\} = -\frac{m}{4R^2}\,\left(1-\phi^\dagger\,\phi\right) \quad \text{and} \quad \left[X^-_{\bar a}\,,\,X^-_{\bar b}\right] = 0 = \left[X^-_a\,,\,X^-_b\right], \tag{5.9}$$

$$m=+1 \qquad \Longrightarrow \qquad X^+_{\bar a}\,\phi - \phi\,X^-_{\bar a} = 0\,, \tag{5.10}$$

$$m=-1 \qquad \Longrightarrow \qquad X^+_{a}\,\phi - \phi\,X^-_{a} = 0\,, \tag{5.11}$$

$$|m|\neq 1 \qquad \Longrightarrow \qquad X^+_{\bar a}\,\phi - \phi\,X^-_{\bar a} = 0 \quad \text{and} \quad X^+_{a}\,\phi - \phi\,X^-_{a} = 0\,. \tag{5.12}$$

Features. The degenerate case $m=0$ corresponds to the trivial reduction (4.1) to $\mathbb{R}^{2n}_\theta$, where the vortex potential disappears and the scalars are covariantly constant in the background of a DUY gauge potential on $\mathbb{R}^{2n}_\theta$. One must also distinguish between $|m|=1$ and $|m|>1$ because these two cases are quite different in nature. For $m=\pm 1$ the gauge potential $\mathcal{A}$ is spherically symmetric (up to a gauge transformation). Eqs. (5.10) and (5.11) are related by the interchanges $z^a \leftrightarrow \bar z^{\bar a}$ and $y \leftrightarrow \bar y$ ($\varphi \leftrightarrow -\varphi$). The field strength component $\mathcal{F}_{\vartheta\varphi}$ is proportional to the angular part of the Dirac monopole (or anti-monopole) field strength tensor. One may regard (5.3) and (5.4) as, respectively, a 'holomorphicity' and 'anti-holomorphicity' condition for ϕ. The commutative analogue of (5.1)–(5.4) was studied in the mathematical literature (see e.g. [35, 36]) under the name of 'coupled vortex equations'. More generally, the situation $|m|>1$ corresponds to m Dirac (anti-)monopoles sitting on top of each other. These configurations are not spherically symmetric. Furthermore, the requirement of 'covariant constancy' (5.5) implies the compatibility conditions

$$F^+_{\mu\nu}\,\phi - \phi\,F^-_{\mu\nu} = 0 \tag{5.13}$$

which are rather restrictive. Nevertheless, we shall see that they can be satisfied nontrivially for the noncommutative instanton configurations to be considered later on.

Finally, we remark that for $n=2$, $m=1$ and $k=1$ the equations (5.1)–(5.3) coincide with the perturbed Seiberg-Witten $\mathrm{U}_+(1)\times\mathrm{U}_-(1)$ monopole equations on $\mathbb{R}^4_\theta$ as considered in [37]. In the commutative limit they reduce to the standard perturbed abelian Seiberg-Witten monopole equations on $\mathbb{R}^4$. The perturbation, i.e. the term $\frac{1}{4R^2}$ in (5.1) and (5.2), is introduced into the Seiberg-Witten equations by hand. In the present context, it arises automatically from the extra space S^2 and the reduction from $\mathbb{R}^4\times S^2$ to $\mathbb{R}^4$.

6. BPS solutions of the Yang-Mills equations on $\mathbb{R}^{2n}_\theta \times S^2$

We are now ready to construct solutions to the generalized coupled vortex equations (5.8)–(5.12), and thus to the DUY equations on $\mathbb{R}^{2n}_\theta \times S^2$, by making use of partial isometries and the (noncommutative) ABS construction.

Explicit solutions. Let us consider the ansatz

$$X_a^{\pm} = \theta_{a\bar{b}}\, T_{N_\pm}\, \bar{z}^{\bar{b}}\, T_{N_\pm}^\dagger \qquad \Longrightarrow \qquad X_{\bar{a}}^{\pm} = \theta_{\bar{a}b}\, T_{N_\pm}\, z^b\, T_{N_\pm}^\dagger\,, \tag{6.1}$$

$$\phi = T_{N_+} T_{N_-}^\dagger \qquad \Longrightarrow \qquad \phi^\dagger = T_{N_-} T_{N_+}^\dagger\,, \tag{6.2}$$

where $T_{N_\pm}$ are $k \times k$ matrices (with operator entries acting on $\mathcal{H}$) possessing the properties

$$T_{N_+}^\dagger T_{N_+} = 1 \qquad \text{while} \qquad T_{N_+} T_{N_+}^\dagger = 1 - P_{N_+}\,, \tag{6.3}$$

$$T_{N_-}^\dagger T_{N_-} = 1 \qquad \text{while} \qquad T_{N_-} T_{N_-}^\dagger = 1 - P_{N_-}\,. \tag{6.4}$$

Here P_{N_+} and P_{N_-} are projectors of rank N_+ and N_-, respectively, on the Fock space $\mathbb{C}^k \otimes \mathcal{H}$, where $\mathcal{H}$ denotes the n-oscillator Fock space. In other words, the operators $T_{N_\pm}$ are partial isometries on $\mathbb{C}^k \otimes \mathcal{H}$. This ansatz directly yields

$$F_{a\bar{b}}^{\pm} = \theta_{a\bar{b}}\, P_{N_\pm} = \frac{1}{2\theta^a}\, \delta_{a\bar{b}}\, P_{N_\pm} \qquad \text{and} \qquad F_{\bar{a}\bar{b}}^{\pm} = 0 = F_{ab}^{\pm}\,, \tag{6.5}$$

$$1 - \phi\, \phi^\dagger = P_{N_+} \qquad \text{and} \qquad 1 - \phi^\dagger\, \phi = P_{N_-}\,. \tag{6.6}$$

Inserting these expressions into our generalized coupled vortex equations (5.8)–(5.12), we find that (5.10)–(5.12) are automatically satisfied. The first two equations reduce to the constraints[5]

$$(5.8) \qquad \Longrightarrow \qquad \frac{1}{\theta^1} + \cdots + \frac{1}{\theta^n} = +\frac{m}{2R^2} \qquad \text{if } N_+ \neq 0\,, \tag{6.7}$$

$$(5.9) \qquad \Longrightarrow \qquad \frac{1}{\theta^1} + \cdots + \frac{1}{\theta^n} = -\frac{m}{2R^2} \qquad \text{if } N_- \neq 0\,. \tag{6.8}$$

For any m and $\theta^a > 0$ (which we assume) these two conditions are incompatible, implying that our ansatz does not allow for BPS configurations with both $N_+ \neq 0$ and $N_- \neq 0$. However, we shall show in the next section that in this case the configuration (6.1), (6.2) still satisfies the *full* Yang-Mills equations on $\mathbb{R}^{2n}_\theta \times S^2$! Yet, in order to solve (5.8) and (5.9) simultaneously one must put either $N_+ = 0$ or $N_- = 0$. For example, if $N_- = 0$ we have $T_{N_-} = 1$ and thus

$$X_a^- = \theta_{a\bar{b}}\, \bar{z}^{\bar{b}} \qquad \Longrightarrow \qquad A_a^- = 0 = A_{\bar{a}}^-$$

$$\Longrightarrow \qquad F^- = 0 \qquad \text{and} \qquad 1 - \phi^\dagger\, \phi = 1 - T_{N_-} T_{N_-}^\dagger = 0\,, \tag{6.9}$$

so that (5.9) becomes an identity and only (6.7) remains. Likewise, $N_+ = 0$ gives viable BPS configurations. Putting $N_+ = 0 = N_-$, however, yields only the vacuum solution $(A{=}0, \phi{=}1)$. Note also that the case $m = 0$ is in conflict with the positivity of all θ^a.

These observations have a natural physical interpretation. The original DUY equations are fixed by the parameters n and k. Our ansatz (4.15) together with (6.1), (6.2) is labelled by the triple (m, N_+, N_-). According to the standard identifications of D-branes as noncommutative solitons [38], for $|m| = 1$ it should describe a collection of N_+ D0-branes

[5] It is possible to generalize the ansatz (6.1) and (6.2), following the discussion of flux tubes and vortices in two dimensions [11, 10]. In this way, the constraints (6.7) and (6.8) may be relaxed [10].

and N_- anti-D0-branes as a bound state (i.e. a vortex-like solution on $\mathbb{R}^{2n}_\theta$) in a system of k $\mathrm{D}(2n) - \overline{\mathrm{D}(2n)}$ brane-antibrane pairs. It is known that such a bound state can only be stable (i.e. possess the BPS property) if either $N_+ = 0$ or $N_- = 0$, which fits perfectly with our findings. Thus, the DUY configurations with $|m| = 1$ obtained from (6.1), (6.2) are stable bound states of $\mathrm{D}(2n) - \overline{\mathrm{D}(2n)}$ pairs containing either N_+ D0-branes or N_- anti-D0-branes, but not both. The D-brane interpretation of our solutions will be elucidated in more detail later on.

Explicit realization of the operators $\boldsymbol{T_{N_\pm}}$. For the operators $T_{N_\pm}$ one may take the matrices T_N from [37] (and references therein). Namely,

$$T_N = (T)^N \qquad \text{with} \qquad T^\dagger = \frac{1}{\sqrt{(\gamma\cdot x)(\gamma\cdot x)^\dagger}}\,\gamma\cdot x\,, \tag{6.10}$$

where $\gamma\cdot x := \gamma_\mu\, x^\mu$, and the $k\times k$ matrices γ_μ are subject to the anti-commutation relations[6]

$${\gamma_\mu}^\dagger\,\gamma_\nu + {\gamma_\nu}^\dagger\,\gamma_\mu = 2\,\delta_{\mu\nu}\,\mathbf{1}_k = \gamma_\mu\,{\gamma_\nu}^\dagger + \gamma_\nu\,{\gamma_\mu}^\dagger\,. \tag{6.11}$$

This implies that

$$\Gamma_\mu = \begin{pmatrix} 0 & {\gamma_\mu}^\dagger \\ -\gamma_\mu & 0 \end{pmatrix} \qquad \text{satisfies} \qquad \Gamma_\mu\,\Gamma_\nu + \Gamma_\nu\,\Gamma_\mu = -2\,\delta_{\mu\nu}\,\mathbf{1}_{2k}\,, \tag{6.12}$$

i.e. they generate the Clifford algebra $C\ell_{2n}$ of the inner product space $(\mathbb{R}^{2n},\delta_{\mu\nu})$. Hence the choice (6.10) restricts us to $k = 2^{n-1}$. Note that for $n = 1 \;\Leftrightarrow\; k = 1$ we have $\gamma_1 = 1$, $\gamma_2 = \mathrm{i}$, which yields

$$T^\dagger = \frac{1}{\sqrt{\bar z z}}\,\bar z = \sum_{k=1}^{\infty} |k{-}1\rangle\langle k|\,, \tag{6.13}$$

and we obtain the standard shift operator $S_N = (T)^N$ in this case.

In general, the operator (6.10) may be regarded as a map

$$T_N : \Delta^-\otimes\mathcal{H} \;\longrightarrow\; \Delta^+\otimes\mathcal{H}\,, \tag{6.14}$$

where $\Delta^\pm \cong \mathbb{C}^k$ are the irreducible chiral spinor modules of dimension $k = 2^{n-1}$ on which the matrices γ_μ act. It is not difficult to see that

$$T_N^\dagger T_N = \mathbf{1}_k\otimes 1 \qquad \text{while} \qquad T_N T_N^\dagger = \mathbf{1}_k\otimes 1 - P_N\,, \tag{6.15}$$

where P_N is a rank-N projector on the space $\mathbb{C}^k\otimes\mathcal{H}$. In particular, the operator $T = T_1$ has no kernel, while $T^\dagger$ has a one-dimensional kernel which is spanned by the vector $|\alpha\rangle\otimes|0,\dots,0\rangle$ where $|\alpha\rangle$ denotes the lowest-weight spinor of $\mathrm{SO}(2n)$. On the other hand, $T^\dagger$ is surjective, $\mathrm{im}(T^\dagger) = \Delta^-\otimes\mathcal{H}$. Consequently,

$$\dim\ker(T_N) = 0 \qquad \text{but} \qquad \dim\,\mathrm{coker}(T_N) = N\,. \tag{6.16}$$

[6] In this part we are more explicit regarding the matrix structure.

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Operators satisfying such conditions are known as Toeplitz operators and generate an algebra called the Toeplitz algebra. The construction (6.10) is known as the (noncommutative) ABS construction [16]. It provides a convenient realization of the operators $T_{N_\pm}$ which we will exploit later on when we analyze explicitly the brane interpretation of our noncommutative multi-instanton solutions. However, even there, only the generic properties of these operators are really important.

Generic form of T_N. The realization (6.10) can be generalized in order to introduce $2nN$ real moduli into the solution, specifying the locations of the N noncommutative solitons in $\mathbb{R}^{2n}$. For this, we will write $T_N = U_1\, U_2 \cdots U_N$ where each $U_\ell^\dagger$ is of the form of $T^\dagger$ in (6.10) but with the coordinates x shifted to $x_\ell := x - b_\ell$. Let us illustrate this strategy on the example of $n = 2 \;\Leftrightarrow\; k = 2$ for the special case of equal noncommutativity parameters $\theta^1 = \theta^2$. We redenote the complex coordinates (z^1, z^2) on $\mathbb{R}^4_\theta$ by (y, z) and represent the 2×2 matrices T and $T^\dagger$ of (6.10) as[7]

$$U = \begin{pmatrix} z & y \\ \bar{y} & -\bar{z} \end{pmatrix} \frac{\mathrm{i}}{r} \qquad \text{and} \qquad U^\dagger = -\frac{\mathrm{i}}{r} \begin{pmatrix} \bar{z} & y \\ \bar{y} & -z \end{pmatrix} \qquad \text{with } r = \sqrt{y\bar{y} + \bar{z}z}\,. \tag{6.17}$$

It is easily checked that they fulfill

$$U^\dagger\, U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{while} \qquad U\, U^\dagger = \begin{pmatrix} 1 - |0,0\rangle\langle 0,0| & 0 \\ 0 & 1 \end{pmatrix} \tag{6.18}$$

and hence the kernel of $U^\dagger$ is spanned by the vector $\binom{1}{0} \otimes |0,0\rangle$. We now introduce the shifted matrices

$$U_\ell^\dagger = -\frac{\mathrm{i}}{r_\ell} \begin{pmatrix} \bar{z}_\ell & y_\ell \\ \bar{y}_\ell & -z_\ell \end{pmatrix} \qquad \text{with } y_\ell := y - b_\ell^y\,, \quad z_\ell := z - b_\ell^z\,, \quad \ell = 1, \ldots, N \tag{6.19}$$

and $r_\ell = \sqrt{y_\ell \bar{y}_\ell + \bar{z}_\ell z_\ell}$. They behave just like $U^\dagger$ in (6.17) except that the kernel is modified according to

$$U_\ell^\dagger \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes |\bar{b}_\ell\rangle = 0 \qquad \text{where} \qquad \bar{y}_\ell|\bar{b}_\ell\rangle = 0 = \bar{z}_\ell|\bar{b}_\ell\rangle\,, \tag{6.20}$$

i.e. $|\bar{b}_\ell\rangle$ is a coherent state depending on the two complex parameters b_ℓ^y and b_ℓ^z.

Consider then the states

$$\|\xi_1\rangle\!\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes |\bar{b}_1\rangle \qquad \text{and} \qquad \|\xi_\ell\rangle\!\rangle := U_1 \cdots U_{\ell-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes |\bar{b}_\ell\rangle \qquad \text{for } \ell = 2, \ldots, N\,. \tag{6.21}$$

Clearly, they are all annihilated by the operator

$$T_N^\dagger := U_N^\dagger \cdots U_\ell^\dagger \cdots U_1^\dagger \tag{6.22}$$

which indeed obeys

$$T_N^\dagger\, T_N = \mathbf{1}_2 \otimes 1 \qquad \text{while} \qquad T_N\, T_N^\dagger = \mathbf{1}_2 \otimes 1 - P_N\,, \tag{6.23}$$

where P_N is the orthogonal projection onto the N-dimensional subspace in $\mathbb{C}^2 \otimes \mathcal{H}$ spanned by the vectors $\|\xi_1\rangle\!\rangle, \ldots, \|\xi_N\rangle\!\rangle$. Similarly, one can introduce operators T_N for $n > 2$ generalizing (6.10).

[7] These matrices can be copied from [39] and references therein.

7. Non-BPS solutions of the Yang-Mills equations on $\mathbb{R}^{2n}_\theta \times S^2$

The DUY equations on $\mathbb{R}^{2n}_\theta \times S^2$ are BPS conditions for the Yang-Mills equations on $\mathbb{R}^{2n}_\theta \times S^2$. Therefore, the configuration (6.1), (6.2) with $T_{N_+} = 1$ or with $T_{N_-} = 1$ produces BPS solutions of the Yang-Mills equations on $\mathbb{R}^{2n}_\theta \times S^2$. We saw that configurations with both N_+ and N_- being non-zero did not satisfy the DUY equations. We shall now demonstrate that for any value of m, N_+ and N_- these still yield non-BPS solutions of the full Yang-Mills equations, with topological charge $\mathcal{Q} = m\,(N_+ - N_-)$.

We recall that

$$\mathcal{A}_a - \theta_{a\bar{b}}\,\bar{z}^{\bar{b}} = X_a^+\,P_+ + X_a^-\,P_- \,, \tag{7.1}$$

$$\mathcal{A}_{\bar{a}} - \theta_{\bar{a}b}\,z^{b} = X_{\bar{a}}^+\,P_+ + X_{\bar{a}}^-\,P_- \,, \tag{7.2}$$

$$\mathcal{A}_\vartheta = (1-\phi)\,P_+\,\frac{\partial P_+}{\partial\vartheta} + (1-\phi^\dagger)\,P_-\,\frac{\partial P_-}{\partial\vartheta}\,, \tag{7.3}$$

$$\mathcal{A}_\varphi = (1-\phi)\,P_+\,\frac{\partial P_+}{\partial\varphi} + (1-\phi^\dagger)\,P_-\,\frac{\partial P_-}{\partial\varphi}\,. \tag{7.4}$$

For the ansatz (6.1) and (6.2), $X_\mu^\pm$ and ϕ are expressed in terms of $T_{N_\pm}$, and we get

$$\mathcal{F}_{a\bar{b}} = \theta_{a\bar{b}}\,\{P_{N_+}\,P_+ + P_{N_-}\,P_-\}\,, \tag{7.5}$$

$$\mathcal{F}_{\vartheta\varphi} = -\mathrm{i}\,\frac{m}{2}\,\sin\vartheta\,\{P_{N_+}\,P_+ - P_{N_-}\,P_-\}\,, \tag{7.6}$$

with all other components of $\mathcal{F}_{ij}$ vanishing. Let us now insert these expressions into the Yang-Mills equations which have the form

$$\frac{1}{\sqrt{g}}\,\partial_i(\sqrt{g}\;\mathcal{F}^{ij}) + [\mathcal{A}_i, \mathcal{F}^{ij}] = 0 \qquad \Longrightarrow \qquad \partial_i(\sqrt{g}\;g^{ik}\,g^{jl}\,\mathcal{F}_{kl}) + \sqrt{g}\;g^{ik}\,g^{jl}\,[\mathcal{A}_i, \mathcal{F}_{kl}] = 0\,, \tag{7.7}$$

where $\sqrt{g} := \sqrt{\det(g_{ij})} = R^2\sin\vartheta$. It is enough to consider the cases $j = c$ and $j = \vartheta, \varphi$ since the case $j = \bar{c}$ can be obtained by hermitian conjugation of (7.7) due to the anti-hermiticity of $\mathcal{A}_i$ and $\mathcal{F}_{ij}$.

For $j = c$, (7.7) reduces to

$$g^{\bar{c}a}\,g^{\bar{b}c}\,\big(\partial_{\bar{c}}\mathcal{F}_{a\bar{b}} + [\mathcal{A}_{\bar{c}}, \mathcal{F}_{a\bar{b}}]\big) = 0 \qquad \Longleftrightarrow \qquad g^{\bar{c}a}\,g^{\bar{b}c}\,[\mathcal{A}_{\bar{c}} - \theta_{\bar{c}b}\,z^b, \mathcal{F}_{a\bar{b}}] = 0\,. \tag{7.8}$$

Substituting (7.2) and (7.5), we see that (7.8) is satisfied due to the identities (4.11), (6.3) and (6.4). In the case $j = \vartheta$, (7.7) simplifies to

$$\partial_\varphi(\sqrt{g}\;g^{\vartheta\vartheta}\,g^{\varphi\varphi}\,\mathcal{F}_{\vartheta\varphi}) + \sqrt{g}\;g^{\vartheta\vartheta}\,g^{\varphi\varphi}\,[\mathcal{A}_\varphi, \mathcal{F}_{\vartheta\varphi}] = 0 \tag{7.9}$$

which turns out to be satisfied identically. Likewise, for $j = \varphi$ one obtains

$$\partial_\vartheta(\sqrt{g}\;g^{\vartheta\vartheta}\,g^{\varphi\varphi}\,\mathcal{F}_{\vartheta\varphi}) + \sqrt{g}\;g^{\vartheta\vartheta}\,g^{\varphi\varphi}\,[\mathcal{A}_\vartheta, \mathcal{F}_{\vartheta\varphi}] = 0 \tag{7.10}$$

which is true as well. Hence, the Yang-Mills equations on $\mathbb{R}^{2n}_\theta \times S^2$ are satisfied, and we have found non-BPS configurations of the form (6.1), (6.2) where both N_+ and N_- are non-zero.

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8. The topological charge

Let us now compute the topological charge (the $(n+1)$-th Chern number) of the above configurations. This calculation is similar to the one in [40]. Namely,

$$\mathcal{F}_{2a-1\,2a} = 2\mathrm{i}\,\mathcal{F}_{a\bar{a}} = -\frac{\mathrm{i}}{\theta^a}\left\{P_{N_+}\,P_+ + P_{N_-}\,P_-\right\}, \tag{8.1}$$

$$\mathcal{F}_{\vartheta\varphi} = -\,\mathrm{i}\,\frac{m}{2}\,\sin\vartheta\left\{P_{N_+}\,P_+ - P_{N_-}\,P_-\right\} \tag{8.2}$$

$$\begin{aligned}\Longrightarrow\quad \mathcal{F}_{12}\,\mathcal{F}_{34}\cdots\mathcal{F}_{2n-1\,2n}\,\mathcal{F}_{\vartheta\varphi} =& (-\mathrm{i})^{n+1}\,\frac{m\,\sin\vartheta}{2\prod_{a=1}^n\theta^a}\left\{P_{N_+}\,P_+ + P_{N_-}\,P_-\right\}^n\times\\ &\times\left\{P_{N_+}\,P_+ - P_{N_-}\,P_-\right\}\\ =&(-\mathrm{i})^{n+1}\,\frac{m\,\sin\vartheta}{2\prod_{a=1}^n\theta^a}\left\{P_{N_+}\,P_+ - P_{N_-}\,P_-\right\}\end{aligned} \tag{8.3}$$

$$\Longrightarrow\quad \mathrm{tr}_{2\times2}(\mathcal{F}_{12}\,\mathcal{F}_{34}\cdots\mathcal{F}_{2n-1\,2n}\,\mathcal{F}_{\vartheta\varphi}) = (-\mathrm{i})^{n+1}\,\frac{m\,\sin\vartheta}{2\prod_{a=1}^n\theta^a}\left\{P_{N_+} - P_{N_-}\right\}, \tag{8.4}$$

where we have used the identities (4.11). It follows that

$$\begin{aligned}\mathrm{tr}_{2\times2}\,\underbrace{\mathcal{F}\wedge\cdots\wedge\mathcal{F}}_{n+1} &= (n+1)!\,\mathrm{tr}_{2\times2}\,\mathcal{F}_{12}\,\mathcal{F}_{34}\cdots\mathcal{F}_{2n-1\,2n}\,\mathcal{F}_{\vartheta\varphi}\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^{2n}\wedge\mathrm{d}\vartheta\wedge\mathrm{d}\varphi\\ &= (n+1)!(-\mathrm{i})^{n+1}\,\frac{m\,(P_{N_+}-P_{N_-})}{2\prod_{a=1}^n\theta^a}\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^{2n}\wedge\sin\vartheta\mathrm{d}\vartheta\wedge\mathrm{d}\varphi\,.\end{aligned} \tag{8.5}$$

With this, the topological charge indeed becomes

$$\begin{aligned}\mathcal{Q} :=& \frac{1}{(n+1)!}\Big(\frac{\mathrm{i}}{2\pi}\Big)^{n+1}\Big(\prod_{a=1}^n 2\pi\theta^a\Big)\ \mathrm{tr}_{\mathbb{C}^k\otimes\mathcal{H}}\int_{S^2}\mathrm{tr}_{2\times2}\,\underbrace{\mathcal{F}\wedge\ldots\wedge\mathcal{F}}_{n+1}\\ =&\left(\frac{\mathrm{i}}{2\pi}\right)^{n+1}(-\mathrm{i})^{n+1}\,\frac{m}{2}\left(\prod_{a=1}^n 2\pi\theta^a\right)\left(\mathrm{Tr}_{\mathbb{C}^k\otimes\mathcal{H}}\,\frac{(P_{N_+}-P_{N_-})}{\prod_{b=1}^n\theta^b}\right)\int_{S^2}\sin\vartheta\mathrm{d}\vartheta\wedge\mathrm{d}\varphi\\ =&\frac{m}{4\pi}\ \mathrm{Tr}_{\mathbb{C}^k\otimes\mathcal{H}}(P_{N_+}-P_{N_-})\int_{S^2}\sin\vartheta\mathrm{d}\vartheta\wedge\mathrm{d}\varphi\\ =&m\mathrm{Tr}_{\mathbb{C}^k\otimes\mathcal{H}}(P_{N_+}-P_{N_-}) = m(N_+-N_-)\,.\end{aligned} \tag{8.6}$$

9. The Yang-Mills functional

Besides the topological charge, it is also instructive to know the value of the (euclidean) action functional on our solutions. For $\mathrm{U}(2k)$ Yang-Mills theory on $\mathbb{R}^{2n}_\theta\times S^2$ this functional has the form

$$E = -\frac{1}{4g_{\mathrm{YM}}^2}\left(\prod_{a=1}^n 2\pi\theta^a\right)\mathrm{Tr}_{\mathcal{H}}\int_{S^2}\mathrm{d}\vartheta\mathrm{d}\varphi\ R^2\sin\vartheta\,\mathrm{tr}_{2k\times2k}\{\mathcal{F}_{ij}\,\mathcal{F}^{ij}\}\,, \tag{9.1}$$

with g_{YM} the Yang-Mills coupling constant. This euclidean action may be interpreted as an energy functional for static Yang-Mills fields in $(2n+2)+1$ dimensions.

For our ansatz (4.6) with a fixed dependence on the spherical coordinates ϑ and φ, this functional can be reduced to an integral ($\mathrm{Tr}_{\mathbb{C}^k\otimes\mathcal{H}}$) over $2n$ dimensions. Substituting (4.22)–(4.26) into (9.1), and performing the integral over S^2 and the trace over $\mathbb{C}^2$ in $\mathbb{C}^{2k}=\mathbb{C}^2\otimes\mathbb{C}^k$, we arrive at

$$E=\frac{2\pi R^2}{g_{\mathrm{YM}}^2}\Big(\prod_{a=1}^{n}2\pi\theta^a\Big)\,\mathrm{Tr}_{\mathbb{C}^k\otimes\mathcal{H}}\Big\{|F^+|^2+|F^-|^2+\frac{m^2+1}{4R^2}\,|D\phi|^2+\frac{m^2+1}{4R^2}\,|(D\phi)^\dagger|^2+ \\ +\frac{m^2}{4R^4}\,(1-\phi\,\phi^\dagger)^2+\frac{m^2}{4R^4}\,(1-\phi^\dagger\,\phi)^2\Big\}\,, \tag{9.2}$$

where we have introduced the customary shorthand notation [41]

$$|F|^2:=\frac{1}{2}\,F_{\mu\nu}{}^\dagger\,F^{\mu\nu}\qquad\text{and}\qquad |D\phi|^2:=(D_\mu\phi)^\dagger\,(D^\mu\phi)\,. \tag{9.3}$$

Note that the relative normalization of the $|F|^2$ and $|D\phi|^2$ terms is not important since it can be changed by rescaling the coordinates [41]. By a Bogomolny type transformation one can show that solutions to the generalized coupled vortex equations (5.1)–(5.5) for m fixed realize absolute minima of the action functional (9.2).[8]

Let us evaluate the action functional (9.2) on our solutions (6.1)–(6.6) to the Yang-Mills equations on $\mathbb{R}^{2n}_\theta\times S^2$, first without assuming the BPS property (6.7) or (6.8). Recall that our ansatz (6.1), (6.2) automatically fulfills

$$D_\mu\phi=0\qquad\text{and}\qquad 1-\phi\,\phi^\dagger=P_{N_+}\,,\quad 1-\phi^\dagger\,\phi=P_{N_-}\,. \tag{9.4}$$

If we assume again for simplicity that $\theta^1=\ldots=\theta^n=:\theta$ then we get

$$|F^+|^2+|F^-|^2=4\,\delta^{a\bar c}\,\delta^{d\bar b}\,\{F^+_{a\bar b}\,F^+_{d\bar c}+F^-_{a\bar b}\,F^-_{d\bar c}\}=\frac{n}{\theta^2}\,\{P_{N_+}+P_{N_-}\}\,. \tag{9.5}$$

Substituting these expressions into (9.2), we finally obtain

$$E=\frac{2\pi R^2}{g_{\mathrm{YM}}^2}\,(2\pi\theta)^n\left(\frac{n}{\theta^2}+\frac{m^2}{4R^4}\right)(N_++N_-)\,. \tag{9.6}$$

The first term yields the tension appropriate to (N_++N_-) D0-branes on a D($2n$)-brane [38] in the Seiberg-Witten decoupling limit, times the area $4\pi R^2$ of the auxiliary sphere S^2. The second term is proportional to the Yang-Mills energy on S^2 of $|m|$ coincident Dirac monopoles. In the BPS case (N_+=0 or N_-=0) we can use the relation

$$|m|\,\theta=2\,n\,R^2 \tag{9.7}$$

from (6.7) or (6.8) to write (9.6) as

$$E_{\mathrm{BPS}}=\frac{1}{2g_{\mathrm{YM}}^2}\,(2\pi)^{n+1}\,\theta^{n-1}\,(n{+}1)\,|m|\,N_\pm\,. \tag{9.8}$$

[8] For a discussion in the undeformed case see [35].

10. Special solutions

The extremal cases $\phi = 0$ and $\phi = 1$ are worth special consideration. We shall now study them in some detail.

Zero monopole sector. Let us first look at the case $\phi = \phi^\dagger = 1$. The generalized coupled vortex equations (5.1)–(5.5) then imply $A^+ = A^- =: A$ and $F^+ = F^- =: F$ as well as

$$\delta^{a\bar{b}}\, F_{a\bar{b}} = 0 \qquad \text{and} \qquad F_{\bar{a}\bar{b}} = 0 = F_{ab}\,, \tag{10.1}$$

which are simply the DUY equations on $\mathbb{R}^{2n}_\theta$ (note that (4.24) implies $\mathcal{F}_{y\bar{y}} = 0 = \mathcal{F}_{\vartheta\varphi}$ if $\phi = 1$). This is equivalent to putting $m = 0$, i.e. the trivial reduction (4.1). Inside the ansatz (6.1), (6.2) we get[9]

$$N_+ = N_- = 0 \qquad \Longrightarrow \qquad P_{N_\pm} = 0 \qquad \Longrightarrow \qquad F^\pm = 0\,. \tag{10.2}$$

This sector can be understood physically as the endpoint of tachyon condensation, wherein the tachyon field ϕ has rolled to its minimum at $\phi = \phi^\dagger = 1$ and all the flux has been radiated away to infinity. Here the D0-branes have been completely dissolved into the D$(2n)$-branes.

Zero tachyon sector. The choice $\phi = 0$ is more interesting since from (4.24) we then have $\mathcal{F}_{\vartheta\varphi} = -\mathrm{i}\,\frac{m}{2}\,\sin\vartheta\,\{P_+ - P_-\}$. For this case (5.1)–(5.5) reduce to

$$\delta^{a\bar{b}}\, F^+_{a\bar{b}} = +\frac{m}{4R^2} \qquad \text{and} \qquad F^+_{\bar{a}\bar{b}} = 0 = F^+_{ab}\,, \tag{10.3}$$

$$\delta^{a\bar{b}}\, F^-_{a\bar{b}} = -\frac{m}{4R^2} \qquad \text{and} \qquad F^-_{\bar{a}\bar{b}} = 0 = F^-_{ab}\,. \tag{10.4}$$

At m=0 this includes the previous case of $\phi = 1$. After switching to the matrix form via (5.6) we obtain

$$\delta^{a\bar{b}}\left[X^+_a\,,\, X^+_{\bar{b}}\right] + \delta^{a\bar{b}}\,\theta_{a\bar{b}} - \frac{m}{4R^2} = 0 \qquad \text{and} \qquad \left[X^+_{\bar{a}}\,,\, X^+_{\bar{b}}\right] = 0 = \left[X^+_a\,,\, X^+_b\right], \tag{10.5}$$

$$\delta^{a\bar{b}}\left[X^-_a\,,\, X^-_{\bar{b}}\right] + \delta^{a\bar{b}}\,\theta_{a\bar{b}} + \frac{m}{4R^2} = 0 \qquad \text{and} \qquad \left[X^-_{\bar{a}}\,,\, X^-_{\bar{b}}\right] = 0 = \left[X^-_a\,,\, X^-_b\right]. \tag{10.6}$$

Since $\phi = 0$ cannot be reached from (6.1), (6.2), we have to look outside this ansatz for solving (10.5) and (10.6). It gives the local maximum of the tachyon potential corresponding to the open string vacuum containing D-branes.

Explicit solutions with $\phi = 0$. Let us restrict ourselves to the abelian case $k = 1$ and simplify matters by taking $\theta^a = \theta$ for all $a = 1, \ldots, n$. We consider the alternative ansatz [23, 25]

$$X^\pm_a = \theta_{a\bar{c}}\, S^\dagger_{l_\pm}\, f_\pm(\mathcal{N}\,)\, \bar{z}^{\bar{c}}\, S_{l_\pm} \qquad \text{and} \qquad X^\pm_{\bar{a}} = \theta_{\bar{a}d}\, S^\dagger_{l_\pm}\, z^d\, f_\pm(\mathcal{N}\,)\, S_{l_\pm}\,, \tag{10.7}$$

[9]However, one may consider (6.1) with $N_+ = N_- \neq 0$ without (6.2) and obtain non-trivial solutions on $\mathbb{R}^{2n}_\theta$ after relaxing the condition $\theta^a > 0$ for all a and allowing at least one of the θ^a to be negative.

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where $f_\pm$ are two functions of the 'total number operator'

$$\mathcal{N} := \frac{1}{2\theta} \sum_{a=1}^{n} z^a \bar{z}^{\bar{a}} \qquad \text{satisfying} \qquad f_\pm(r) = 0 \qquad \text{for } r \le l_\pm{-}1\,. \tag{10.8}$$

The shift operators $S_{l_\pm}$ in (10.7) are defined to obey

$$S^\dagger_{l_\pm}\, S_{l_\pm} = 1 \quad \text{while} \quad S_{l_\pm}\, S^\dagger_{l_\pm} = 1 - \Pi_{l_\pm} \quad \text{with} \quad \Pi_{l_\pm} := \sum_{|k|\le l_\pm - 1} |k_1,\ldots,k_n\rangle\langle k_1,\ldots,k_n|\,, \tag{10.9}$$

where $|k| := k_1 + \ldots + k_n$. Note that

$$S^\dagger_{l_\pm}\, \Pi_{l_\pm} = \Pi_{l_\pm}\, S_{l_\pm} = 0 \qquad \text{and} \qquad f_\pm(\mathcal{N}\,)\, \Pi_{l_\pm} = \Pi_{l_\pm}\, f_\pm(\mathcal{N}\,) = 0\,, \tag{10.10}$$

and $S^\dagger_{l_\pm}$ projects all states with $|k| < l_\pm$ out of $\mathcal{H}$.

One easily sees that (10.7) fulfills the homogeneous equations in (10.5) and (10.6). Remembering that $\theta_{a\bar{b}} = -\theta_{\bar{b}a} = \frac{1}{2\theta^a}\,\delta_{a\bar{b}} = \frac{1}{2\theta}\,\delta_{a\bar{b}}$, we also obtain

$$\begin{aligned}
\big[X^\pm_a\,,\, X^\pm_{\bar{b}}\big] &= \theta_{a\bar{c}}\,\theta_{\bar{b}d}\, S^\dagger_{l_\pm}\Big\{ f_\pm(\mathcal{N}\,)\, \bar{z}^{\bar{c}}\, (1{-}\Pi_{l_\pm})\, z^d\, f_\pm(\mathcal{N}\,) \,- \\
&\qquad\qquad\qquad - z^d\, f_\pm(\mathcal{N}\,)\, (1{-}\Pi_{l_\pm})\, f_\pm(\mathcal{N}\,)\, \bar{z}^{\bar{c}} \Big\}\, S_{l_\pm} \\
&= -\frac{1}{4\theta^2}\, \delta_{a\bar{c}}\, \delta_{d\bar{b}}\, S^\dagger_{l_\pm} \Big\{ f^2_\pm(\mathcal{N}\,)\, \bar{z}^{\bar{c}} z^d - f^2_\pm(\mathcal{N}{-}1)\, z^d \bar{z}^{\bar{c}} \Big\} S_{l_\pm}
\end{aligned} \tag{10.11}$$

with the help of the identities $\bar{z}^{\bar{c}}\,\Pi_{l_\pm} = \Pi_{l_\pm - 1}\,\bar{z}^{\bar{c}}$ where $\Pi_0 := 0$ as well as

$$\bar{z}^{\bar{c}}\, f_\pm(\mathcal{N}\,) = f_\pm(\mathcal{N}{+}1)\, \bar{z}^{\bar{c}} \qquad \text{and} \qquad z^d\, f_\pm(\mathcal{N}\,) = f_\pm(\mathcal{N}{-}1)\, z^d\,. \tag{10.12}$$

Substituting (10.11) into (10.5) and (10.6), we employ

$$\delta_{\bar{c}d}\, z^d \bar{z}^{\bar{c}} = 2\theta\, \mathcal{N} \qquad \text{and} \qquad \delta_{\bar{c}d}\, \bar{z}^{\bar{c}} z^d = 2\theta\, (\mathcal{N} + n) \tag{10.13}$$

to find the conditions

$$\begin{aligned}
0 &= \delta^{a\bar{b}}\, \big[X^\pm_a\,,\, X^\pm_{\bar{b}}\big] + \delta^{a\bar{b}}\,\theta_{a\bar{b}} \mp \frac{m}{4R^2} \\
&= -\frac{1}{2\theta}\, S^\dagger_{l_\pm} \Big\{ f^2_\pm(\mathcal{N}\,)\, (\mathcal{N}{+}n) - f^2_\pm(\mathcal{N}{-}1)\, \mathcal{N} \Big\}\, S_{l_\pm} + \frac{n}{2\theta} \mp \frac{m}{4R^2} \\
&= \frac{1}{2\theta}\, S^\dagger_{l_\pm} \Big\{ \mathcal{N}\, f^2_\pm(\mathcal{N}{-}1) - (\mathcal{N}{+}n)\, f^2_\pm(\mathcal{N}\,) + n \Big(1 \mp \frac{m\,\theta}{2nR^2} \Big) \Big\}\, S_{l_\pm}
\end{aligned} \tag{10.14}$$

on the operators $f_\pm$. These recursions are solved by[10]

$$f^2_\pm(\mathcal{N}\,) = \Big(1 \mp \frac{m\,\theta}{2nR^2} \Big) \Big(1 - \frac{\mathcal{Q}_\pm\; n!}{(\mathcal{N}{+}1)\cdots(\mathcal{N}{+}n)} \Big)\, \Theta(\mathcal{N}{-}l_\pm{+}1)\,, \tag{10.15}$$

where

$$\mathcal{Q}_\pm := \frac{l_\pm (l_\pm{+}1)\cdots(l_\pm{+}n{-}1)}{n!}\,. \tag{10.16}$$

[10] Θ denotes the Heaviside step function. It may be replaced by $1{-}\Pi_{l_\pm}$. The ambiguity at $\mathcal{N} = l_\pm{-}1$ is irrelevant here because then the prefactor vanishes.

Assuming that $1-\frac{m\,\theta}{2nR^2}>0$, we can take a positive square root to get

$$X_a^{\pm}=\frac{1}{2\theta^{\pm}}S_{l_{\pm}}^{\dagger}\left(1-\frac{\mathcal{Q}_{\pm}\;n!}{(\mathcal{N}+1)\cdots(\mathcal{N}+n)}\right)^{\frac{1}{2}}\Theta(\mathcal{N}-l_{\pm}+1)\,\delta_{a\bar{c}}\,\bar{z}^{\bar{c}}\,S_{l_{\pm}}\;, \tag{10.17}$$

where we abbreviated

$$\theta^{\pm}=\frac{\theta}{\sqrt{1\mp\frac{m\,\theta}{2nR^2}}}\;. \tag{10.18}$$

The $m{=}0$ case (equivalent to $\phi{=}1$) is also covered by these solutions. Note that for $m{\neq}0$ the solutions (10.17) coincide with those obtained in [23, 25] if one assigns different noncommutativity parameters θ^+ and θ^- to the worldvolumes of D($2n$)-branes and D($2n$)-antibranes, respectively. Then the field strengths $F^{\pm}(\theta^{\pm})$ on $\mathbb{R}^{2n}_{\theta^{\pm}}$ obtained from (10.17) will have finite topological charges $\mathcal{Q}_{\pm}$ given by (10.16), as calculated in [23, 25]. The interesting idea of introducing distinct noncommutativity parameters on multiple (coincident) D-branes (generated by different magnetic fluxes on their worldvolumes [42]) was discussed in [43]. This proposal gains support from our zero-tachyon BPS solutions (10.17) which carry oppositely oriented magnetic fluxes on branes versus antibranes.

11. Multi-instanton K-cycles

In the remainder of this paper we will work towards clarifying the D-brane interpretations of the multi-instanton solutions that we have found. This will be done by illustrating that the solution (6.1), (6.2) can be very naturally obtained via a construction in K-homology. Passing from analytic to topological K-homology will then provide a worldvolume picture of these solutions in which the brane interpretations become manifest. We shall find that, for $|m|=1$, a configuration of k D($2n$+2)-branes and k D($2n$+2)-antibranes wrapping a common sphere S^2 with the monopole field is equivalent to k D($2n$)-branes and k D($2n$)-antibranes, i.e. the DUY equations on $\mathbb{R}^{2n}\times S^2$ are equivalent to generalized vortex equations on $\mathbb{R}^{2n}$. This means that instantons on $\mathbb{R}^{2n}_\theta\times S^2$ are the spherical extensions of vortices which are points in $\mathbb{R}^{2n}_\theta$. Then, the solutions to the generalized vortex equations produces (with $k{=}2^{n-1}$) N_+ D0-branes and N_- D0-antibranes in the worldvolume $\mathbb{R}^{2n}_\theta$. But from the point of view of the initial brane-antibrane system on $\mathbb{R}^{2n}_\theta\times S^2$, they are spherical N_+ D2-branes and N_- D2-antibranes. For $|m|>1$ this equivalence ceases to hold and requires us to introduce the notion of a "D-operation" using another standard construction in K-theory. In this case the solutions correspond instead to D0-branes in the initial D($2n$+2) $-$ $\overline{\text{D}(2n{+}2)}$ brane-antibrane system on $\mathbb{R}^{2n}_\theta\times S^2$ which carry additional moduli labelling their position in the auxiliary sphere S^2. In this section we will describe the pertinent K-theoretic (co)cycles, and then use them in the next section to illustrate these features.

The monopole cocycle. The existence of instanton solutions with non-trivial flux relies crucially on the presence of a non-trivial Dirac monopole configuration on the auxiliary space S^2, which is also a crucial ingredient of the K-theoretic construction. The K-theory

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charge group of the total space $\mathbb{R}^{2n} \times S^2$ can be calculated through the suspension isomorphism to give

$$\mathsf{K}^0(\mathbb{R}^{2n} \times S^2) = \mathsf{K}^0(S^2) = \mathbb{Z} \oplus \mathbb{Z}\,, \tag{11.1}$$

where throughout K-theory with compact support is always understood (i.e. $\mathbb{R}^{2n}$ is understood topologically via its one-point compactification as S^{2n}). The Bott generator of the reduced K-theory group $\widetilde{\mathsf{K}}^0(S^2) = \mathbb{Z}$ may be represented by the line bundle $\mathcal{L}_{\rm D}$ over $\mathbb{C}P^1 \cong S^2$, which classifies the Dirac monopole. Regarded as the non-trivial map between pairs of bundles over $\mathbb{R}^2$, i.e. as the class of a virtual bundle $[\mathcal{L}_{\rm D}, \overline{\mathcal{L}_{\rm D}}; \nu_{\rm D}] \in \widetilde{\mathsf{K}}^0(S^2)$, the generator is determined by the standard (commutative) ABS configuration in codimension 2,

$$(\nu_{\rm D})_{(r,\varphi)} = {\rm e}^{{\rm i}\varphi}\,, \tag{11.2}$$

where (r, φ) are polar coordinates on $\mathbb{R}^2$. This is the canonical generator of $\pi_1(S^1) = \mathbb{Z}$ and is the usual commutative tachyon field configuration in codimension 2.

Bott periodicity then induces an isomorphism

$$\alpha : \mathsf{K}^0(\mathbb{R}^{2n}) \otimes \mathsf{K}^0(S^2) \xrightarrow{\ \cup\ } \mathsf{K}^0(\mathbb{R}^{2n} \times S^2) \longrightarrow \mathsf{K}^0(\mathbb{R}^{2n})\,, \tag{11.3}$$

where the first map is the cup product and we use the fact that all K-theory groups in the present case are freely generated. Explicitly, evaluated on a virtual pair $[E^+, E^-; \tau] \in \mathsf{K}^0(\mathbb{R}^{2n})$, with τ the tachyon field isomorphism on the rank k bundles $E^- \to E^+$ at infinity, the isomorphism (11.3) is given by

$$\big[E^+\,,\,E^-\,;\,\tau\big] \ \longmapsto\ \alpha\big[E^+ \otimes (\mathcal{L}_{\rm D} \oplus \overline{\mathcal{L}_{\rm D}})\,,\, E^- \otimes (\mathcal{L}_{\rm D} \oplus \overline{\mathcal{L}_{\rm D}})\,;\, \tau \bullet \nu_{\rm D}\big] \tag{11.4}$$

with

$$\tau \bullet \nu_{\rm D} = \begin{pmatrix} \tau \otimes \mathbf{1}_2 & \mathbf{1}_k \otimes \nu_{\rm D}^\dagger \\ \mathbf{1}_k \otimes \nu_{\rm D} & -\tau^\dagger \otimes \mathbf{1}_2 \end{pmatrix}\,. \tag{11.5}$$

The peculiar form of this product owes to the periodicity properties of the underlying Clifford algebra in the ABS construction, i.e. in the standard decomposition of the gamma-matrices (6.12) in terms of lower-dimensional ones. For further details, see [18]. This product isomorphism generated by the monopole bundle over S^2 will play an important role in what follows. The topological equivalence $\mathsf{K}^0(\mathbb{R}^{2n} \times S^2) = \mathsf{K}^0(\mathbb{R}^{2n})$ will essentially imply the equivalence of the brane-antibrane systems on $\mathbb{R}^{2n} \times S^2$ and $\mathbb{R}^{2n}$.

Analytic tachyon K-cycles. The configurations described by (6.2) correspond to noncommutative solitons. Written in terms of the noncommutative ABS configuration (6.10), they correspond in fact to noncommutative tachyons [38, 16]. To make this statement precise we will now show that these configurations are directly related to ordinary commutative ABS configurations and can, for our purposes, be simply treated by using the classical constructions. This will also explicitly demonstrate that the K-theory of the commutative and noncommutative configurations are the same, which paves the way to our eventual worldvolume description. This is accomplished via a standard K-theoretic mapping between analytic (noncommutative) and topological (commutative) descriptions [16, 44]. The basic idea is that, viewed in the noncommutative space $\mathbb{R}^{2n}_\theta \times S^2$, the vortex configuration ϕ is an element of the algebra $C(S^2) \otimes \mathcal{K}(\mathbb{C}^k \otimes \mathcal{H})$, where $C(S^2)$ is the algebra of continuous complex-valued functions on S^2 and $\mathcal{K}$ denotes the algebra of compact operators. By Morita equivalence, the corresponding analytic K-theory classes live in

$$\mathsf{K}_0\big(C(S^2) \otimes \mathcal{K}(\mathbb{C}^k \otimes \mathcal{H})\big) = \mathsf{K}_0\big(C(S^2)\big) = \mathsf{K}^0\big(S^2\big)\,, \tag{11.6}$$

which is the noncommutative version of the suspension isomorphism (11.1). In this analytic setting, Bott periodicity is the equivalence

$$\mathsf{K}_0\big(C(\mathbb{R}^{2n}) \otimes \mathrm{C}\ell_2\big) = \mathsf{K}_0\big(C(\mathbb{R}^{2n})\big)\,. \tag{11.7}$$

The crux of this identification is the fact that the noncommutative ABS configuration $T = T_1$ in (6.10) defines a cycle of the analytic K-homology group $\mathsf{K}_0^{\rm a}(\mathbb{R}^{2n})$ [16, 45]. With $\mathcal{H}^\pm = \Delta^\pm \otimes \mathcal{H}$, it is a bounded Fredholm operator $T : \mathcal{H}^- \to \mathcal{H}^+$. Let $\mathcal{B}(\mathcal{H}^\pm)$ denote the algebras of bounded linear operators on the separable Hilbert spaces $\mathcal{H}^\pm$, and let $\rho^\pm : C(\mathbb{R}^{2n}) \to \mathcal{B}(\mathcal{H}^\pm)$ be representations of the algebra of functions on $\mathbb{R}^{2n}_\theta$ by pointwise, diagonal multiplication (representing $\mathcal{H}$ in the Bargmann polarization, for example). Then for each $f \in C(\mathbb{R}^{2n})$, the operator T is "almost" compatible with the two representations in the sense that

$$T\,\rho^-(f) - \rho^+(f)\,T ~\in~ \mathcal{K}(\mathcal{H}^+ \oplus \mathcal{H}^-)\,. \tag{11.8}$$

In addition, we have

$$T\,T^\dagger - \mathbf{1} ~\in~ \mathcal{K}(\mathcal{H}^+) \qquad \text{and} \qquad T^\dagger\,T - \mathbf{1} ~\in~ \mathcal{K}(\mathcal{H}^-)\,. \tag{11.9}$$

Such a quintuple $(\mathcal{H}^+, \mathcal{H}^-, \rho^+, \rho^-; T)$ is called an (even) Fredholm module. The abelian group of stable homotopy classes of Fredholm modules is the analytic K-homology group $\mathsf{K}_0^{\rm a}(\mathbb{R}^{2n}) = \mathsf{K}^0(C(\mathbb{R}^{2n}))$. The charge of a class $[\mathcal{H}^+, \mathcal{H}^-, \rho^+, \rho^-; T] \in \mathsf{K}_0^{\rm a}(\mathbb{R}^{2n})$ is the analytic index of T,

$$\mathrm{index}(T) = \dim\ker(T) - \dim\,\mathrm{coker}(T)\,. \tag{11.10}$$

The connection with the commutative K-theory description now proceeds with the observation that the same K-homology class comes from a Dirac operator on the space $\mathbb{R}^{2n}$ [44]. For this, let us consider the chiral spinor bundles $\mathbf{\Delta}^\pm \to \mathbb{R}^{2n}$ of rank $k = 2^{n-1}$, and let $\not{D} = -\mathrm{i}\gamma \cdot \partial : C^\infty(\mathbb{R}^{2n}, \mathbf{\Delta}^-) \to C^\infty(\mathbb{R}^{2n}, \mathbf{\Delta}^+)$ be the corresponding Dirac operator on the spaces of smooth spinors on $\mathbb{R}^{2n}$. By completing these vector spaces using the induced inner product from the metric of $\mathbb{R}^{2n}$, we may view $\not{D}$ as an unbounded linear operator

$$\not{D} : L^2(\mathbb{R}^{2n}, \mathbf{\Delta}^-) \longrightarrow L^2(\mathbb{R}^{2n}, \mathbf{\Delta}^+)\,. \tag{11.11}$$

With $\rho^\pm : C(\mathbb{R}^{2n}) \to \mathcal{B}(L^2(\mathbb{R}^{2n}, \mathbf{\Delta}^\pm))$ the representations by pointwise multiplication as above, it follows that $(L^2(\mathbb{R}^{2n}, \mathbf{\Delta}^+), L^2(\mathbb{R}^{2n}, \mathbf{\Delta}^-), \rho^+, \rho^-; \not{D}/|\not{D}|)$ is a Fredholm module. The corresponding class $[\not{D}] \in \mathsf{K}_0^{\rm a}(\mathbb{R}^{2n})$ depends only on the original Dirac operator $\not{D} : C^\infty(\mathbb{R}^{2n}, \mathbf{\Delta}^-) \to C^\infty(\mathbb{R}^{2n}, \mathbf{\Delta}^+)$.

The particularly noteworthy aspect of this correspondence [44] is that the index (11.10) of the noncommutative ABS configuration coincides with that of the corresponding Dirac operator,

$$\mathrm{index}(T) = \mathrm{index}(\not{D}) \,. \tag{11.12}$$

On the other hand, the index of $\not{D} : C^\infty(\mathbb{R}^{2n}, \boldsymbol{\Delta}^-) \to C^\infty(\mathbb{R}^{2n}, \boldsymbol{\Delta}^+)$ is just the virtual dimension (zeroth Chern number) of the index class $[\ker(\not{D}), \mathrm{coker}(\not{D}); \not{D}] \in \mathsf{K}^0(\mathbb{R}^{2n})$, so that

$$\mathrm{index}(T) = \mathrm{ch}_0\big(\ker(\not{D}) \ominus \mathrm{coker}(\not{D})\big) \,. \tag{11.13}$$

This coincides with the K-theory charge of the Bott class $[\boldsymbol{\Delta}^+, \boldsymbol{\Delta}^-; \mu] \in \mathsf{K}^0(\mathbb{R}^{2n})$ given by the ABS construction, where $\mu_x : \Delta^- \to \Delta^+$ is Clifford multiplication by $x \in \mathbb{R}^{2n}$,

$$\mu_x = \frac{\gamma \cdot x}{|x|} \,, \tag{11.14}$$

and it is the generator of $\pi_{2n-1}(\mathrm{U}(k)) = \mathbb{Z}$. In this way, the analytic index of the noncommutative tachyon operator (11.10) coincides with the winding number of the classical ABS tachyon field (11.14) at infinity in $\mathbb{R}^{2n}$.

It can be shown that all classes in $\mathsf{K}_0^{\mathrm{a}}(\mathbb{R}^{2n})$ arise in this way [44]. In particular, the above construction also identifies the Poincaré duality isomorphism $\mathsf{K}_0^{\mathrm{a}}(\mathbb{R}^{2n}) \cong \mathsf{K}^0(\mathbb{R}^{2n})$ between K-homology and K-theory through the map $[\mathcal{H}^+, \mathcal{H}^-, \rho^+, \rho^-; T] \mapsto [\boldsymbol{\Delta}^+, \boldsymbol{\Delta}^-; \mu]$. It is in this way that we will be able to use a commutative description in what follows, with the ordinary ABS field (11.14). Everything that we have said also easily generalizes to the higher degree configurations $T_N = (T)^N$ used in (6.1) and (6.2). By considering the Dirac operator corresponding to the twisted spinor bundles $\boldsymbol{\Delta}^\pm \otimes ((\mathcal{L}_{\mathrm{D}})^m \oplus (\overline{\mathcal{L}_{\mathrm{D}}})^m)$, we can further twist the Fredholm operator (6.2) by the class $[(\mathcal{L}_{\mathrm{D}})^m, (\overline{\mathcal{L}_{\mathrm{D}}})^m; (\nu_{\mathrm{D}})^m]$ of the degree m monopole bundle over S^2. Given the monopole projectors $P_\pm$ in (4.10), we choose a partial isometry ς on $\mathbb{C}^2$ obeying

$$\varsigma^\dagger\, \varsigma = \mathbf{1}_2 \qquad \text{while} \qquad \varsigma\, \varsigma^\dagger = \mathbf{1}_2 - P_+ \,. \tag{11.15}$$

Then from the relations $P_{N_\pm} = 1 - T_{N_\pm} T_{N_\pm}^\dagger$ we may compute the corresponding index as

$$\begin{aligned} \mathrm{index}\big(\phi \bullet (\varsigma)^m\big) &= \dim\ker \begin{pmatrix} \phi \otimes \mathbf{1}_2 & \mathbf{1}_k \otimes (\varsigma^\dagger)^m \\ \mathbf{1}_k \otimes (\varsigma)^m & -\phi^\dagger \otimes \mathbf{1}_2 \end{pmatrix} - \dim\ker \begin{pmatrix} \phi^\dagger \otimes \mathbf{1}_2 & \mathbf{1}_k \otimes (\varsigma^\dagger)^m \\ \mathbf{1}_k \otimes (\varsigma)^m & -\phi \otimes \mathbf{1}_2 \end{pmatrix} \\ &= m \dim\ker(T_{N_-}^\dagger) - m\, \dim\ker(T_{N_+}^\dagger) \\ &= m\, \mathrm{Tr}_{\mathbb{C}^k \otimes \mathcal{H}}(P_{N_-}) - m\, \mathrm{Tr}_{\mathbb{C}^k \otimes \mathcal{H}}(P_{N_+}) = m(N_- - N_+) \,. \end{aligned} \tag{11.16}$$

The K-theory charge (11.16) of the noncommutative soliton configuration (6.1), (6.2) thereby coincides with the topological charge $-\mathcal{Q}$ computed in the Yang-Mills theory on $\mathbb{R}^{2n}_\theta \times S^2$ in (8.6).

12. Worldvolume interpretation

Using the analysis of the previous section we are now ready for the D-brane interpretation of the multi-instanton solutions. For our purposes a D-brane will be specified by a triple (W, E, σ), where $\mathbb{R}^1 \times W \cong \mathbb{R}^1 \times \mathbb{R}^{2n}$ is the brane worldvolume regarded as a spin^c

submanifold of the ambient space $\mathbb{R}^1 \times X \cong \mathbb{R}^1 \times W \times S^2$, and $\sigma : W \hookrightarrow X$ is its embedding.[11] $E \to W$ is the complex Chan-Paton vector bundle with connection (extended trivially along the time direction $\mathbb{R}^1$). More precisely, in the following construction we will only need the stable isomorphism class $\xi \in \mathsf{K}^0(W)$ of the Chan-Paton bundle. Virtual bundles will also be encountered later on and correspond to (unstable) brane-antibrane configurations wrapping W.

With a suitable equivalence relation put on (W, E, σ), to be described in detail below, the set of all equivalence classes of D-branes $[W, E, \sigma]$ generates an abelian group called the topological K-homology group $\mathsf{K}_0^{\rm t}(X)$ [16, 45]. The connection with the noncommutative tachyon configurations of the previous section is provided by the isomorphism

$$\kappa : \mathsf{K}_0^{\rm t}(X) \longrightarrow \mathsf{K}_0^{\rm a}(X) \tag{12.1}$$

of abelian groups which is defined as follows. Using the given data, we form the Hilbert spaces $\mathcal{H}_E^{\pm} = L^2(W, \boldsymbol{\Delta}^{\pm}(W) \otimes E)$, where $\boldsymbol{\Delta}^{\pm}(W) \to W$ are the chiral spinor bundles of rank $k = 2^{n-1}$ induced by the spin^c structure on W. Let $\not{D}_E : \mathcal{H}_E^- \to \mathcal{H}_E^+$ be the corresponding twisted Dirac operator. Then, as explained before, this construction produces an analytic K-cycle which defines a class $[\not{D}_E] \in \mathsf{K}_0^{\rm a}(W)$. Under the push-forward $\sigma_* : \mathsf{K}_0^{\rm a}(W) \to \mathsf{K}_0^{\rm a}(X)$ this induces an analytic K-homology class on the ambient space X and the map (12.1) is thereby defined by

$$\kappa[W, E, \sigma] = \sigma_*[\not{D}_E]\,. \tag{12.2}$$

It can be shown that this map is well-defined and invertible [44]. This follows essentially from our previous remarks about the equivalence between the commutative and noncommutative descriptions.

There are three equivalence relations that need to be put on the D-brane (W, E, σ) [45]. They are bordism (continuous deformations of the brane worldvolume W and of the Chan-Paton bundle E), direct sum (gauge symmetry enhancement for coincident D-branes), and vector bundle modification (dielectric effect). We claim that for $|m| = 1$ the multi-instanton solutions (6.1), (6.2) provide a physical realization of this latter relation, so we shall study it in some detail.

Starting from (W, E, σ) we can build another triple which represents the same K-homology class by using the Baum-Douglas clutching construction [44]. For this, let $I = W \times \mathbb{C}$ denote the trivial line bundle over W, and let $H \to W$ be a spin^c vector bundle of rank 2. We use the induced metric on H to define the ball and sphere bundles $\boldsymbol{B}(H)$ and $\boldsymbol{\Sigma}(H)$ over H, which are also spin^c vector bundles. In particular, the sphere bundle of $H \oplus I$ can be constructed by gluing together two copies $\boldsymbol{B}(H)_{\pm}$ of the ball bundle of H using the identity map along their common boundary $\boldsymbol{\Sigma}(H) = \partial \boldsymbol{B}(H)_{\pm}$ to give

$$\boldsymbol{\Sigma}(H \oplus I) = \boldsymbol{B}(H)_+ \cup_{\boldsymbol{\Sigma}(H)} \boldsymbol{B}(H)_-\,. \tag{12.3}$$

[11] In this context we are working with type IIA branes. More generally, we can take $W = \mathbb{R}^{2n} \times W_r$ where W_r is any spin^c manifold of dimension r (with all fields independent of the coordinates on W_r). For odd r this then allows for type IIB branes and degree 1 K-groups. For notational simplicity we only work explicitly with the case $W = \mathbb{R}^{2n}$, as it captures the essential features and is easily generalized.

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Note that locally the sphere bundle $\boldsymbol{\Sigma}(H\oplus I)$ is isomorphic to our ambient space $X = W\times S^2$ and so, with a slight abuse of notation, we will identify these two manifolds. Let

$$\pi:\boldsymbol{\Sigma}(H\oplus I)\longrightarrow W \tag{12.4}$$

be the corresponding bundle projection.

Denote by $\boldsymbol{\Delta}^{\pm}(H)$ the pull-backs of the chiral spinor bundles $\boldsymbol{\Delta}^{\pm}(W)\to W$ to H. As usual, the vector bundle map induced by Clifford multiplication

$$\mu:\boldsymbol{\Delta}^{-}(H)\longrightarrow\boldsymbol{\Delta}^{+}(H) \tag{12.5}$$

is an isomorphism off the zero section of $H\to W$, i.e. at "infinity". We may now define a vector bundle over (12.3) by putting $\boldsymbol{\Delta}^{\pm}(H)$ over $\boldsymbol{B}(H)_{\pm}$. Since the line bundles $\boldsymbol{\Delta}^{\pm}(H)\to\boldsymbol{B}(H)$ are isomorphic over $\boldsymbol{\Sigma}(H)$ by Clifford multiplication (12.5), we can glue them together along the sphere bundle $\boldsymbol{\Sigma}(H)$ using the transition function μ to define

$$\Xi=\boldsymbol{\Delta}^{+}(H)\cup_{\mu|_{\boldsymbol{\Sigma}(H)}}\boldsymbol{\Delta}^{-}(H)\,. \tag{12.6}$$

This is the virtual bundle $\Xi=\boldsymbol{\Delta}^{+}(H)\ominus\boldsymbol{\Delta}^{-}(H)$ which defines a class $[\boldsymbol{\Delta}^{+}(H),\boldsymbol{\Delta}^{-}(H);\mu]\in\mathsf{K}^0(X)$. For each point $w\in W$ on the D-brane worldvolume, $\pi^{-1}(w)$ is a two-dimensional sphere S^2, and $\Xi|_{\pi^{-1}(w)}$ yields the Bott class $[\boldsymbol{\Delta}^{+},\boldsymbol{\Delta}^{-};\mu]$ which generates $\widetilde{\mathsf{K}}^0(S^2)=\mathbb{Z}$. This class is the same as that of the monopole cocycle $[\mathcal{L}_{\rm D},\overline{\mathcal{L}_{\rm D}};\nu_{\rm D}]$ introduced previously.

Vector bundle modification is then the equivalence relation which can be formulated as the equality between topological K-homology classes of the D-branes

$$[W\,,\,\xi\,,\,\sigma]=[X\,,\,\Xi\otimes\pi^*\xi\,,\,\sigma\circ\pi] \tag{12.7}$$

for any (virtual) Chan-Paton bundle $\xi\in\mathsf{K}^0(W)$. In particular, if the left-hand side of (12.7) corresponds to the class of k $\mathrm{D}(2n)-\overline{\mathrm{D}(2n)}$ pairs wrapping $W\cong\mathbb{R}^{2n}$, i.e. $\xi=E^+\ominus E^-$ with $\mathrm{ch}_0(E^{\pm})=k$, then the right-hand side corresponds to k $\mathrm{D}(2n{+}2)-\overline{\mathrm{D}(2n{+}2)}$ pairs wrapping $X\cong\mathbb{R}^{2n}\times S^2$. This is simply the equivalence between instantons on $\mathbb{R}^{2n}\times S^2$ and vortices on $\mathbb{R}^{2n}$ that we encountered before.

The key point now is that for a suitable choice of tachyon field on the left-hand side of (12.7), the right-hand side coincides with the K-homology class of the multi-instanton solution (6.1), (6.2). For this, we recall that the classical ABS configuration (11.14) for $x\in\mathbb{R}^{2n}$ represents the class of the noncommutative ABS operator T in (6.10). It follows that the noncommutative tachyon field (6.2) corresponds to the K-theory class

$$\xi=\left[E^+\,,\,E^-\,;\,(\mu)^{N_+}\,(\mu^\dagger)^{N_-}\right] \tag{12.8}$$

over the worldvolume $W\cong\mathbb{R}^{2n}$. On the other hand, the relation (12.7) equates the resulting K-homology class with that defined by

$$\Xi\otimes\pi^*\xi=\left[\pi^*E^+\otimes(\mathcal{L}_{\rm D}\oplus\overline{\mathcal{L}_{\rm D}})\,,\,\pi^*E^-\otimes(\mathcal{L}_{\rm D}\oplus\overline{\mathcal{L}_{\rm D}})\,;\,\widetilde{\mu}\right] \tag{12.9}$$

over the ambient space $X \cong \mathbb{R}^{2n} \times S^2$, where

$$\widetilde{\mu} = \begin{pmatrix} \pi^*\,(\mu)^{N_+}\,(\mu^\dagger)^{N_-}\,\sigma^* \otimes \mathbf{1}_2 & \mathbf{1}_k \otimes \nu_{\rm D}^\dagger \\ \mathbf{1}_k \otimes \nu_{\rm D} & -\pi^*\,(\mu)^{N_-}\,(\mu^\dagger)^{N_+}\,\sigma^* \otimes \mathbf{1}_2 \end{pmatrix} . \tag{12.10}$$

The charge of the class (12.9) is given by (11.16) for $|m| = 1$, and it thereby describes, through the standard process of tachyon condensation on the unstable system of k D$(2n+2)$-branes and k D$(2n+2)$-antibranes wrapping X, a configuration of spherical N_+ D2-branes and N_- D2-antibranes. On the left-hand side of (12.7), these are instead D0-branes arising from vortices left over from condensation in the transverse space $\mathbb{R}^{2n}$. Thus the vortices become instantons with worldvolume S^2. The worldvolume interpretation of the multi-instanton solutions corresponding to monopole charges $|m| > 1$ is not so straightforward. In this case one needs to twist $\pi^*\xi$ instead with the higher-degree monopole bundle $(\mathcal{L}_{\rm D})^m$, as in (11.16). This twisting no longer preserves the topological K-homology classes. We shall see below how to overcome this difficulty and hence interpret these solutions K-theoretically.

D-brane charge. We will now derive a geometrical formula for the topological charge of the multi-soliton configuration in terms of the standard characteristic classes associated with the configuration of D-branes. For this, we regard the original worldvolume embedding as the closed embedding $\sigma : W \to \mathbf{\Sigma}(H \oplus I)$, which has normal bundle $\boldsymbol{B}(H)_+ - \mathbf{\Sigma}(H) \cong H$. The corresponding Gysin homomorphism $\sigma_! : \mathsf{K}^0(W) \to \mathsf{K}^0(X)$ is then defined such that the following diagram commutes:

$$\begin{array}{ccc} \mathsf{K}^0(W) & \longrightarrow & \mathsf{K}_0^{\rm t}(W) \\ \sigma_! \downarrow & & \downarrow \sigma_* \\ \mathsf{K}^0(X) & \longleftarrow & \mathsf{K}_0^{\rm t}(X) \end{array} \tag{12.11}$$

where the horizontal arrows denote Poincaré duality isomorphisms. It may be conveniently represented in a way that refers only to K-theory groups through the sequence of maps

$$\sigma_! : \mathsf{K}^0\big(W\big) \;\longrightarrow\; \mathsf{K}^0\big(H\big) \;\longrightarrow\; \mathsf{K}^0\big(X\,,\,\boldsymbol{B}(H)_-\big) \;\longrightarrow\; \mathsf{K}^0\big(X\big)\,, \tag{12.12}$$

where the first arrow is the Thom isomorphism of H, the second one is the excision isomorphism, and the last map is restriction induced by the inclusion $(X, \emptyset) \hookrightarrow (X, \boldsymbol{B}(H)_-)$. From (12.12) one can then show [46] that $\Xi \otimes \pi^*\xi = \sigma_!\xi \oplus \pi^*\xi$ in $\mathsf{K}^0(X)$, and the summand $\pi^*\xi$ can be eliminated by using the gauge symmetry enhancement relation when defining the topological K-homology group. In other words, we can replace the vector bundle modification relation (12.7) by

$$[W\,,\,\xi\,,\,\sigma] = [X\,,\,\sigma_!\xi\,,\,\sigma \circ \pi]\,. \tag{12.13}$$

The Chern character of a K-cycle (W, ξ, σ) in the homology of the space X is given by [44] the push-forward σ_* of the Poincaré dual of the characteristic class ${\rm ch}(\xi) \wedge {\rm Td}(TW)$

in the rational cohomology of the worldvolume W, where ch denotes the (graded) Chern character and $\mathrm{Td}(TW)$ is the Todd class of the tangent bundle of W. By using (11.12), (12.2), (12.13) and the ordinary Atiyah-Singer index theorem (expressed within the framework of K-homology [44]), we thereby arrive at the topological formula

$$\mathcal{Q}_{|m|=1} = -\mathrm{index}(\phi) = -\int_W \sigma^*\big(\mathrm{ch}(\sigma_!\xi) \wedge \mathrm{Td}(TX)\big) . \tag{12.14}$$

This formula illustrates two important features. First of all, it expresses the fact that the equivalence of the charges in the commutative and noncommutative theories is simply the equality of the analytic and topological indices. Secondly, it expresses the topological charges of the multi-instantons on $\mathbb{R}^{2n}_\theta \times S^2$ as the standard formula [18] for the charge of a D-brane (W, ξ, σ) in terms of characteristic classes of the ambient space X. In particular, it explicitly illustrates how the instanton number on $\mathbb{R}^{2n} \times S^2$ is equivalent to a vortex charge on $\mathbb{R}^{2n}$.

D-operations. The solutions we have obtained for $|m| > 1$ possess many properties different from those at $|m| = 1$. For instance, in the BPS case, instead of a simple holomorphicity or anti-holomorphicity constraint as in (5.3) or (5.4) for $|m| = 1$, the tachyon field ϕ is required to be covariantly constant as in (5.5). Furthermore, a major difference between the $|m| = 1$ and $|m| > 1$ cases lies in their behaviour under the SU(2) isometry group of the two-sphere. As we have discussed, for $|m| = 1$ the gauge potential has a generalized (i.e. up to gauge transformations) SU(2) invariance. This invariance leads to the equivalence of instantons on $\mathbb{R}^{2n} \times S^2$ and vortices on $\mathbb{R}^{2n}$ since no additional moduli arise from the S^2 dependence in this case. From the K-homology point of view, this is equivalent to the vector bundle modification argument given above. On the other hand, for $|m| > 1$ the configuration is no longer homogeneous over the S^2, so that rotations produce moduli which are not spurious because they cannot be killed by gauge transformations.

Therefore, our $|m| > 1$ solutions should not be interpreted as D2-branes in a D($2n$+2) brane-antibrane system, but rather as D0-branes inside D($2n$+2) brane-antibrane pairs which may carry additional moduli indicating their location not only in $\mathbb{R}^{2n}$ but also in S^2. For $|m| = 1$ the S^2 moduli are unphysical because they are gauge artifacts, but this is not so for $|m| > 1$. Thus the D($2n$) and D($2n$+2) brane systems are not equivalent, and the vector bundle modification argument given above breaks down. This moduli dependence is particularly clear from the form of the field strength tensors (7.5) and (7.6) which show how the monopole degrees of freedom interlace with the field theory degrees of freedom on $\mathbb{R}^{2n}_\theta$. For example, consider the zero tachyon sector where (10.3) and (10.4) are satisfied. The m-monopole flux on S^2 defines the degree m line bundle $(\mathcal{L}_{\mathrm{D}})^m \to S^2$. Then eqs. (10.3) and (10.4) imply that the degrees (first Chern numbers) of the rank k complex vector bundles E^+ and E^- on the branes and antibranes are m and $-m$, respectively (we implicitly use here an appropriate compactification of $\mathbb{R}^{2n}$ to make the total magnetic flux finite). Thus the abelian fluxes over $\mathbb{R}^{2n}$ and S^2 are correlated, and they lead to well-defined solutions with finite action and topological charge. In the remainder of this section we propose a K-theoretic interpretation which naturally explains this flux correlation and which provides the appropriate extension of the framework described above to the case $|m| > 1$.

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The mapping which mixes fluxes in the manner described above will be referred to as a "D-operation".[12] An operation in K-theory is a natural map

$$\Psi : \mathsf{K}^0(W) \longrightarrow \mathsf{K}^0(W) \tag{12.15}$$

defined for every worldvolume W, which is also natural in W. In this sense an operation is a symmetry of K-theory. The only operations in complex K-theory which are ring homomorphisms, i.e. which obey $\Psi(\xi \oplus \xi') = \Psi(\xi) \oplus \Psi(\xi')$ and $\Psi(\xi \otimes \xi') = \Psi(\xi) \otimes \Psi(\xi')$, are the Adams operations. For each $m \in \mathbb{Z}$, they are given by

$$\Psi^m(E) = Q_m(\wedge^1 E, \dots, \wedge^m E)\,, \tag{12.16}$$

where $\wedge^p E$ denotes the (class of the) p-th exterior power of the Chan-Paton bundle $E \to W$. Here $Q_0 := 1$ and Q_m, $m \geq 1$ is the m-th Newton polynomial which expresses the symmetric function $\sum_a (u_a)^m$ as the unique polynomial of the elementary symmetric functions e_p of $u_1, \dots, u_m$, i.e.

$$Q_m(e_1, \dots, e_m) = \sum_{a=1}^m (u_a)^m \qquad \text{with } e_p = \sum_{a_1 < \dots < a_p} u_{a_1} \cdots u_{a_p}\,. \tag{12.17}$$

For example,

$$Q_1(e_1) = e_1\,, \qquad Q_2(e_1, e_2) = (e_1)^2 - 2e_2\,, \qquad Q_3(e_1, e_2, e_3) = (e_1)^3 - 3e_1 e_2 + 3e_3\,, \tag{12.18}$$

and so on. For $m < 0$, (12.16) is defined with the arguments $\wedge^1 \overline{E}, \dots, \wedge^{|m|} \overline{E}$. The Adams operations (12.16) may be conveniently represented through the generating function defined by

$$\sum_{m=1}^\infty (-t)^{m-1}\ \Psi^m(E) = \frac{\mathrm{d}}{\mathrm{d}t} \ln\Big(1 + \sum_{p=1}^\infty t^p\ \wedge^p E\Big)\,. \tag{12.19}$$

For our purposes the importance of the Adams operations stems from two important properties that they possess (see [47] for further details). First of all, if $\mathcal{L}$ is (the class of) any line bundle, then

$$\Psi^m(\mathcal{L}) = \mathcal{L}^m\,. \tag{12.20}$$

Secondly, if $W = \mathbb{R}^{2n}$ and $\xi \in \mathsf{K}^0(W)$, then

$$\Psi^m(\xi) = m^n\, \xi := \underbrace{\xi \oplus \dots \oplus \xi}_{m^n}\,. \tag{12.21}$$

Let us now apply this symmetry to the pertinent bundle $\xi \otimes \mathcal{L}_{\mathrm{D}}$ over $\mathbb{R}^{2n} \times S^2$ used in (12.8)–(12.10). Using the fact that the Adams operation is a ring homomorphism on K-theory, for a fixed monopole charge $m \in \mathbb{Z}$ we find

$$\Psi^m(\xi \otimes \mathcal{L}_{\mathrm{D}}) = \Psi^m(\xi) \otimes \Psi^m(\mathcal{L}_{\mathrm{D}}) = (m^n\, \xi) \otimes (\mathcal{L}_{\mathrm{D}})^m\,. \tag{12.22}$$

[12]This is not to be confused with the Steenrod square cohomology operations used in [48].

Now let $\mathbb{Q}_m$ be the subring of $\mathbb{Q}$ consisting of fractions with denominator a power of m. We then define an operation

$$\widetilde{\Psi}^m : \mathsf{K}^0(W) \longrightarrow \mathsf{K}^0(W) \otimes \mathbb{Q}_m \tag{12.23}$$

by $\widetilde{\Psi}^m = \Psi^m / m^{n-1}$ for $W = \mathbb{R}^{2n}$. From (12.22) it then follows that the corresponding Chern character is modified to

$$\mathrm{ch}\big(\widetilde{\Psi}^m (\xi \otimes \mathcal{L}_{\mathrm{D}})\big) = m \ \mathrm{ch}\big(\xi \otimes (\mathcal{L}_{\mathrm{D}})^m\big) . \tag{12.24}$$

Consequently, the mapping $\xi \mapsto \widetilde{\Psi}^m(\xi)$ induces the appropriate charge twisting as required in (8.6) and (11.16) by effectively increasing the number of D0-branes left over after tachyon condensation by a factor of $|m|$. This immediately leads to the physical implications of operations in K-theory. We propose that the appropriate multi-instanton K-cycle obtained from the reduction $\mathbb{R}^{2n} \times S^2 \to \mathbb{R}^{2n}$ is $(W, \widetilde{\Psi}^m(\xi), \sigma)$, where $\mathrm{ch}_0(\xi) = N_- - N_+$ as before. From the point of view of the original noncommutative gauge theory on $\mathbb{R}^{2n}_\theta \times S^2$, it is the cycle $(X, \widetilde{\Psi}^m(\Xi \otimes \pi^*\xi), \sigma \circ \pi)$, with the appropriate higher-degree bundle (12.22) as specified by our original ansatz (4.15) (note that the Adams operation obeys the naturality condition $\Psi^m\, \pi^* = \pi^*\, \Psi^m$). Now, however, since the Adams operation Ψ^m is not generally an isomorphism on K-theory (but rather only a symmetry), the vector bundle modification relation (12.7) breaks down and we no longer have the equivalence of the brane-antibrane systems on $\mathbb{R}^{2n}$ and $\mathbb{R}^{2n} \times S^2$. Note that for $|m| = 1$, one has $\widetilde{\Psi}^1(\xi) = \xi$, and the class of the D-brane remains as before. In this way the (modified) Adams operation $\xi \mapsto \widetilde{\Psi}^m(\xi)$ takes into account the non-spurious monopole moduli dependence of the D-brane state for $|m| > 1$, and sharply captures the differences between the $|m| = 1$ and $|m| > 1$ cases. More generally, we propose that the usual descent relations among D-branes in string theory can be naturally understood as a consequence of the symmetries of K-theory.

13. Summary and discussion

In this paper we have constructed explicit solutions to the Yang-Mills equations on the noncommutative space $\mathbb{R}^{2n}_\theta \times S^2$ with gauge group $\mathrm{U}(2k)$ and arbitrary magnetic flux over the S^2. We have obtained generic BPS solutions corresponding to multi-instantons on $\mathbb{R}^{2n}_\theta \times S^2$ by solving the noncommutative Donaldson-Uhlenbeck-Yau equations. They are uniquely determined by $\mathrm{U}(k)$ vortex configurations on $\mathbb{R}^{2n}_\theta$. The BPS conditions on these solutions are given by (6.7) or (6.8) and relate the noncommutativity parameters θ^a, the radius R of the two-sphere, and the topological charge m of the monopole bundle in the space $\mathbb{R}^{2n}_\theta \times S^2$. We have also obtained non-BPS solutions to the full Yang-Mills equations on $\mathbb{R}^{2n}_\theta \times S^2$ which likewise arise from a reduction to $\mathbb{R}^{2n}_\theta$. Generally, the solutions are labelled by three integers (N_+, N_-, m) (with $N_+{=}0$ or $N_-{=}0$ in the BPS case), and carry other moduli associated with the locations of the noncommutative solitons in $\mathbb{R}^{2n}$. The moduli space of these solutions is (up to discrete symmetries)

$$\mathcal{M}(N_+, N_-, m) = \mathbb{C}^{nN_+} \times \mathbb{C}^{nN_-} \times \mathbb{C}P^1 \subset \mathrm{Gr}_{N_+}(\infty) \times \mathrm{Gr}_{N_-}(\infty) \times \mathbb{C}P^1 , \tag{13.1}$$

where $\mathrm{Gr}_N(\infty)$ is the infinite-dimensional grassmannian manifold which parametrizes the rank-N projectors on the Hilbert space $\mathbb{C}^k \otimes \mathcal{H}$, and the $\mathbb{C}P^1$ factor parametrizes the position in S^2. This manifold includes the special solutions which can be interpreted as extrema of a string-inspired potential for the Higgs field configurations on $\mathbb{R}^{2n}_\theta$. Note that neither the topological charge (8.6) nor the Yang-Mills energy (9.6) depend on the translational moduli in $\mathbb{R}^{2n}$.

Since $\mathrm{Gr}_N(\infty)$ is also the classifying space for rank N complex vector bundles, to each multi-instanton solution living in (13.1) we may associate a (virtual) bundle $\xi_0 \otimes (\mathcal{L}_{\mathrm{D}})^m$, where $\xi_0 = E_0^+ \ominus E_0^-$ and $E_0^\pm \to \mathbb{R}^{2n}$ are bundles of rank $N_\pm$. We interpret this as the Chan-Paton bundle of N_+ D2-branes and N_- D2-antibranes wrapping S^2, inside a system of $k = 2^{n-1}$ D$(2n{+}2)$ brane-antibrane pairs wrapping $\mathbb{R}^{2n} \times S^2$. This interpretation follows from the choice of ansatz for the $u(2k)$-valued gauge potential $\mathcal{A}$ on $\mathbb{R}^{2n}_\theta \times S^2$. Generically, such a gauge field would live on $2k$ coincident D-branes, all of which carry the same orientation. However, with a choice of multi-monopole solution on S^2, the gauge symmetry is broken down to $u(2k) \to u(k) \oplus u(k)$. In the ansatz (4.15) we then couple the $u(k)$ gauge connection A^+ with the monopole projector and A^- with the anti-monopole projector, leading to opposite monopole charges between pairs of D-branes and hence opposite orientations. In other words, due to the presence of a non-trivial monopole field on the two-sphere, one can alter the orientation of a D-brane such that different branes from the initial set of $2k$ branes wrap with opposite orientation around the S^2.

An interesting aspect of our solutions is the rather drastic differences between the cases $|m| = 1$ and $|m| > 1$. This is even apparent in the reduced form (9.2) of the Yang-Mills functional on $\mathbb{R}^{2n}_\theta$. When $m^2 = 1$, the relative normalization between the kinetic and potential energy terms for the Higgs field ϕ is precisely of the form needed to rewrite the action as the Yang-Mills functional of a superconnection, as is expected of the energy of a basic brane-antibrane system (see e.g. [49]). In this case BPS solutions correspond to equating the action to the topological charge $\mathcal{Q}$, which implies immediately the generalized vortex equations that we found from the Donaldson-Uhlenbeck-Yau equations on $\mathbb{R}^{2n}_\theta \times S^2$. On the other hand, when $m^2 > 1$, the action (9.2) does not yield the standard superconnection form of a brane-antibrane energy functional. BPS solutions in this case require not only the generalized vortex equations to be satisfied, but also the more rigid equation $|D\phi|^2 = 0$. A heuristic way to understand the differences here from the point of view of the effective Yang-Mills-Higgs system obtained after reduction is to look at the case $n = 1$, $k = 1$ with $A^+ = -A^-$ $(B = 0)$ in the commutative limit $\theta = 0$. Then the action (9.2) describes the standard Ginzburg-Landau model of superconductivity [41]. The case $m^2 = 1$ corresponds to the BPS case when there are no forces between the vortices. On the other hand, for $m^2 > 1$, vortices attract each other and there is a bound state with finite energy and topological charge (but it is not a BPS solution as the second order Yang-Mills-Higgs equations are solved). Using this analogy it is tempting to speculate that in these instances some analog of the Meissner efffect (complete expulsion of magnetic flux) occurs in the combined brane-monopole system on $\mathbb{R}^{2n}_\theta \times S^2$. From the K-theory point of view, the multi-instanton solutions on $\mathbb{R}^{2n}_\theta \times S^2$ correspond to a symmetry in K-theory acting on the initial brane-antibrane Chan-Paton bundle. For $m^2 = 1$ this symmetry is simply the identity map and

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it directly yields the equivalence of the brane-antibrane systems on $\mathbb{R}^{2n}_\theta$ and $\mathbb{R}^{2n}_\theta \times S^2$ in topological K-homology, but this is not so in the cases $m^2 > 1$. It would be interesting to understand in more detail the D-brane physics of the noncommutative solitons constructed in this paper and to further clarify the role of the Adams operations in their description.

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